Students’ Errors in Setting up Difference Quotients and Connections to Their Conceptions of Function

BY

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THESIS

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1919 - 2010
ACKNOWLEDGMENTS

An African proverb tells us that it takes a village to raise a child. I would like to thank the following people in my village for their unwavering support. Thank you:

**Mom** for telling me, “College is the best time of your life,” so often that I decided never to leave! I know that every decision you ever made was for the benefit of your children. Thank you for teaching me about *strength*.

**Janet** for the many hours dedicated to getting me outta here. Without your support and dedication, I may never have graduated. Thank you for teaching me about *patience*.

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Illustrating APOS conceptions applied to point-wise composition of \( f(x) = x^2 \) and \( g(x) = x + 1 \) at \( x = 2 \).

Illustrating APOS conceptions applied to uniform function composition \( f(x) = x^2 \) and \( g(x) = x + 1 \) at \( x = 2 \).

Sample of Correct Student Work for Question 1.

Sample of Incorrect Student Work for Question 1.

Scores for Question 1 by Lecture.

Sample of Correct Student Work.

Sample of Incorrect Student Work.

Scores for Question 2 by Lecture.

Sample of Correct Student Work for Question 3.

Two Samples of Incorrect Student Work for Question 3.

Scores for Question 3 by Lecture.

Sample of Correct Student Work for Question 4.

Sample of Chain Rule Student Work for Question 4.

Sample of Incorrect Student Work for Question 4.

Scores for Question 4 by Lecture.

Interviewee Question Scores.

Lian’s Misuse of the equal sign for the function \( f(x) = x^2 + 1 \).

Lian’s Misuse of the equal sign for the function \( f(x) = x + 2 \).
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Sample 1 shown to Interviewees

Number of Function Representations
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<td>AMS</td>
<td>American Mathematical Society</td>
</tr>
<tr>
<td>CTAN</td>
<td>Comprehensive TeX Archive Network</td>
</tr>
<tr>
<td>TUG</td>
<td>TeX Users Group</td>
</tr>
<tr>
<td>UIC</td>
<td>University of Illinois at Chicago</td>
</tr>
<tr>
<td>UICTHESI</td>
<td>Thesis formatting system for use at UIC.</td>
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<td>DQ(h)</td>
<td>(\frac{f(x + h) - f(x)}{h})</td>
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<tr>
<td>DQ(a)</td>
<td>(\frac{f(x) - f(a)}{x - a})</td>
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SUMMARY

Although finding the limits of the difference quotients in the definitions of the derivative is troubling for many students, a difficulty that preceded this confusion was observed: students were not able to correctly set up the difference quotients as required in the definitions. The purpose of this study is to uncover student errors in setting up the difference quotients and to discuss what these errors reveal about students’ thinking of functions, and evaluations of the difference quotients. At the end of the study, a framework that aggregates criteria used (by past studies and this study) to assign student membership into a function conception category will be produced in an attempt to move towards a systematic classification of students’ cognitive processes. Implications from this study can inform teaching practices and curriculum development, by helping students connect difference quotient evaluation with function composition.
CHAPTER 1

INTRODUCTION

1.1 Problem

Research shows that calculus students generally have difficulty finding the derivative of a function using the limit definition (Zandieh, 2000):

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]  

While the concept of limit is one that is troubling to many students, I noticed a difficulty that precedes this confusion with limits: students were not able to correctly evaluate \( f(x + h) \) in the difference quotient as required in the definition.

To document the prevalence of student difficulty with finding \( f(x + h) \), I examined student work on the following question of a calculus final exam:\(^1\)

Find \( \frac{dy}{dx} \) by using the limit definition of the derivative where \( f(x) = 3x^2 + x \).

On the 142 exams examined, 27 students did not follow the directions to use the limit definition to answer the problem. The students in this group used the power rule, gave no answer, or gave incomprehensible answers. The rest of the students all used Equation 1.1. On these remaining

\(^1\)Calculus for Business
115 exams, 31 students—27%—failed to correctly give $f(x + h)$, which is the crucial first step of completing this problem. This is an alarming statistic that warrants further investigation.

1.2 Goals

A goal of this study is to identify the patterns of error students make when setting up the difference quotient in Equation 1.1. It follows naturally to study the errors made with the difference quotient in another limit definition for derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

so analysis of student errors on this evaluation was also included.

Identifying the patterns of error is crucial in providing insight into students’ conceptions about the difference quotients and about functions in general. This study seeks to gain a deeper understanding of these errors and what they reveal about students’ thinking. Exposing the trends in student thinking as a revealed in discussions of these errors alerts educators to reasons why students struggle with these important concepts and points to areas of content that need special attention. In addition, disclosing the conversations about these errors may inform curricular development towards ways of strengthening students’ understanding of the difference quotients and functions.

In an attempt to categorize students’ thinking of function, the theory of “Action, Process, Object, Schema” (APOS Theory) was initially used. A review of the literature on APOS shows a
mismatch in the intended and the enacted theory, causing difficulty in data analysis (see Section 2.1).

To overcome this difficulty, this study seeks to extend the previous work done with APOS by contributing data from student interviews for the definitions provided by the theory. This contribution is in the form of data analysis tables (see, for example Table XXIX) intending to aid researchers in assigning students into function, composition, and $f(x + h)$ conception categories. These tools will be created from criteria used in this study and in past studies.

The remainder of the study will denote the difference quotients in Equation 1.2 and Equation 1.1 as:

\[
\frac{f(x + h) - f(x)}{h}, \quad (DQ_h)
\]

and

\[
\frac{f(x) - f(a)}{x - a}. \quad (DQ_a)
\]

DQ is used to refer to both difference quotients.

1.3 Research Questions

Evaluation of $f(x + h)$ as required in Equation 1.1 is considered a composition problem (Horvath, 2011), $f(g(x)) = f(x + h)$ where $g(x) = x + h$. Therefore, in addition to studying students’ conceptions of function and DQ evaluations, it is also appropriate to investigate student thinking about composition in the conversations about student errors. The application of function composition is relevant not only in the limit definition provided in Equation 1.1, but also in applications of the chain rule, and methods of integration in the first, second and third courses of calculus. The importance of function composition extends to differential equations
and partial differential equations because understanding the concepts in these courses requires mastery of these skills.

This study identifies where students are struggling with the difference quotients of the limit definitions of the derivative, what these patterns of struggle look like, and what educators can do to mitigate them. Furthermore, this study produces data analysis tables to assist with categorizing students’ conceptions of function, composition, and \( f(x + h) \). To this end, the following research questions will be explored:

1. What errors do students make when evaluating \( DQ_a \), \( DQ_h \), and routine function compositions? Can these errors be categorized in a way that is independent of the type of function (e.g. monomial, rational, trigonometric functions)?

2. What do students’ errors reveal about their conceptions of function, function composition, and \( DQ_a \) and \( DQ_h \) evaluation?

3. How can students’ conceptions of function, function composition, and \( f(x + h) \) discovered from this study contribute to APOS theory?

1.4 Methodology

This study was conducted in two main phases. The first phase is dedicated to answering Research Question 1. As such, a survey design was used to reveal errors in quiz questions involving students’ \( DQ_a \) and \( DQ_h \) evaluations and compositions similar to \( f(x + h) \). Analysis in this phase aims to report on error trends. T-tests and ANOVAs were conducted to compare
means, while Pearson product-moment correlations were used to measure the strength and directions of relationships between the scores of each respective quiz question.

Research Question 2 was addressed in the second phase using the qualitative method of case study. Four errors chosen from the first phase were studied in-depth through interviews designed to elicit students' understandings of function, composition, and DQ evaluations.

To begin answering Research Question 3, a review of the literature on APOS was conducted to aggregate definitions of action, process, and object provided by both the creators and other users of the theory. Unified definitions were created from these definitions and matched with data from student interviews.

1.5 **Organization of the Dissertation**

The APOS theoretical framework is discussed in detail in Chapter 2, where a review of the APOS literature shows the need to update the original definitions posited by creators of the theory. Chapter 2 also aggregates definitions found in the literature for understanding students' conceptions of function and function composition. A review of the broader literature on functions, studies which used APOS, and research on students and function composition follows in Chapter 3.

In Chapter 4, the errors found from analyzing student quizzes are presented. Common error categories from across quiz questions are presented along with findings from statistical analyses of the students' quiz scores. In-depth case study analyses of four of these error categories are found in Chapter 5.
Chapter 6 presents the data analysis tables created from the student data from both Chapter 5 and the review of the APOS literature in Chapter 2. It also shows how these data analysis tables were created and how they can be used to assess students’ conceptions of functions, composition, and $f(x+h)$. Implications of the study are presented in Chapter 7 and concluding statements in Chapter 8.
CHAPTER 2

THEORETICAL FRAMEWORK

Action, Process, Object, Schema (APOS) Theory is a widely used theoretical approach to assess student understanding of mathematical concepts, particularly the function concept. This theory extends Piaget’s work on reflective abstraction in children’s learning to post-secondary students’ cognitive developments. It has been tested empirically by the original creators and outside researchers through cyclic iterations of student observations and theory refinement. As a result, the definitions for action, process, object, and schema have evolved from the researchers’ original declarations. As the research community has become more stable in understanding the epistemology of mathematical concepts through empirical data, the definitions have also settled.

The words action, process, object, and schema are used to describe the stages one must pass through in order to obtain full understanding of mathematical concepts. They can be used to describe a student’s degree of understanding (e.g., S/he has an action conception of function), or to describe the mathematical concept itself (e.g., The function is viewed as an object when it is acted on). Section 2.1 will give an overview of the general definitions provided by the theory. These definitions are applied to different mathematical concepts such as function, and function composition in Section 2.2 and Section 2.3.
2.1 General Theory

According to APOS theory, the action conception is the first stage students must pass through in order to move into a full understanding of a mathematical concept. If students display evidence of understanding a mathematical concept as an action, then they are said to have an action conception of that concept.

Table I shows the evolution of the action definition. The action definition began with the sufficient condition of mental or physical manipulation of objects (Breidenbach et al., 1992), to a somewhat external transformation (Asiala et al., 1996), to an essentially external transformation (Dubinsky and Mcdonald, 2002), and ends with the necessary condition of an explicit (either verbal, or written) transformation (Dubinsky et al., in press). In the theory’s infancy, evidence of a mental transformation was enough to be considered an action conception. However, in the modern theory, the steps of the transformation must be explicitly carried out in order to be considered an action and a mental transformation are reserved for a process conception.\(^1\)

“When the total action can take place entirely in the mind of the subject, or just be imagined as taking place, without necessarily running through all of the specific steps,” then (Breidenbach et al., 1992, p. 249) say that the learner has moved to a process conception. The evolution of the definition of a process conception is not as drastic compared with that of the action definition, so it has not changed much. The presence of a process conception has two specifications:

\(^1\)Examples of the theory as applied to functions and composition are detailed in the next two sections.
**TABLE I: THE EVOLUTION OF ACTION DEFINITION IN APOS THEORY**

<table>
<thead>
<tr>
<th>Definition</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>• “static conception” that requires “repeatable mental or physical</td>
<td>(Breidenbach et al., 1992, p. 251) (pre-treatment)</td>
</tr>
<tr>
<td>manipulation of objects”...“one step at a time”</td>
<td>(Breidenbach et al., 1992, p. 278) (post-treatment)</td>
</tr>
<tr>
<td>• “transforms objects [but need] for an explicit recipe or formula that</td>
<td>(Dubinsky and Harel, 1992, p. 51)</td>
</tr>
<tr>
<td>describes the transformation [which is relatively external to the</td>
<td>(Asiala et al., 1996, p. 7)</td>
</tr>
<tr>
<td>thinking of the subject]...in a step-by-step manner related only by the</td>
<td>(Dubinsky and Mcdonald, 2002, p. 276)</td>
</tr>
<tr>
<td>recipe, and not by any relationships that exist in the mind of the subject”</td>
<td>(Ed Dubinsky, e-mail, January 2, 2012.)</td>
</tr>
<tr>
<td>• “thinking about and discussing a procedure in terms of the individual</td>
<td>(Dubinsky et al., in press)</td>
</tr>
<tr>
<td>steps is what would be expected from an action response”</td>
<td></td>
</tr>
<tr>
<td>• “transformation of objects which is perceived by the individual as</td>
<td></td>
</tr>
<tr>
<td>being at least somewhat external. That is, an individual whose</td>
<td></td>
</tr>
<tr>
<td>understanding of a transformation is limited to an action conception can</td>
<td></td>
</tr>
<tr>
<td>carry out the transformation only by reacting to external cues that give</td>
<td></td>
</tr>
<tr>
<td>precise detail on what steps to take”</td>
<td></td>
</tr>
<tr>
<td>• “transformation of objects perceived by the individual as essentially</td>
<td></td>
</tr>
<tr>
<td>external and as requiring, either explicitly or from memory, step-by-step</td>
<td></td>
</tr>
<tr>
<td>instructions on how to perform the operation”</td>
<td></td>
</tr>
<tr>
<td>• “The steps of the transformation must be actually carried out or</td>
<td></td>
</tr>
<tr>
<td>written down. There can be no generality.”</td>
<td></td>
</tr>
<tr>
<td>• “set of step-by-step instructions performed explicitly to transform</td>
<td></td>
</tr>
<tr>
<td>physical or mental objects”</td>
<td></td>
</tr>
</tbody>
</table>
1. the transformation exists entirely in the mind of the individual (Asiala et al., 1996; Breidenbach et al., 1992; Dubinsky et al., in press; Dubinsky and Mcdonald, 2002), and

2. the individual does not require external stimuli (Asiala et al., 1996; Breidenbach et al., 1992; Dubinsky and Mcdonald, 2002).

An important aspect of the process conception is that it begins with the same procedural view of the transformation as in the action conception, except that instead of explicitly performing the procedure, it is completely imagined in the learner’s mind. Another distinguishing feature of a process response is that can be very general, unlike an action which requires specificity. (Ed Dubinsky, e-mail, January 2, 2012).

Along with the two requirements above, the theory also specifies two indications of a process conception: students can reverse the steps of transformation (Asiala et al., 1996; Breidenbach et al., 1992), or they can combine it with other processes to form new processes.\(^1\) If there is evidence students can do these things, they can be classified as having a process conception, but they are not required to in order to display possession of a process conception. For example, it is not necessary to include students’ knowledge of derivatives in order to classify their conception of function, particularly if they have not learned about derivatives. However, if they have learned about derivatives, then their conceptions about derivatives can indicate their level of function conception.

\(^1\)This is Piaget’s definition of \textit{coordination} (Dubinsky, 1991).
In the next stage, a process is reified (Sfard, 1992) or encapsulated (Breidenbach et al., 1992) into an object. Historically, the object conception has four requirements:

1. must come from a process (Breidenbach et al., 1992; Dubinsky and Mcdonald, 2002),

2. awareness of the process as a totality (Asiala et al., 1996; Dubinsky and Mcdonald, 2002),

3. realization that the process can be acted on with previously established actions or processes (Asiala et al., 1996; Dubinsky and Mcdonald, 2002), and

4. explicit performance of a previously established action or process on the process (Asiala et al., 1996; Breidenbach et al., 1992; Ed Dubinsky, e-mail, January 2, 2012).

The last statement requires a process to be acted on in order to be considered an object. However, consider the numeral 17; it can be viewed in totality without being acted on.\(^1\) Similarly, students can view a function in totality as a set of transformed domain values without acting on it. If a student displays this understanding, s/he has surpassed the process category, but has not quite reached the object conception. It is unclear where to place this student in his/her understanding. Recently, to address this very issue, (Dubinsky et al., in press) created a new stage between the process and object conceptions.

\(^1\)This is similar to Piaget’s idea of object permanence. That is, an object exists regardless if one can see, touch, or hear it, or in this case, perform mathematical operations.
This new transition stage is taken directly from the second requirement of the object conception and is called the Totality (TOT) stage. Totality was created in the context of the infinite decimal expansion of 0.\(\bar{9}\), so its definition is very specific: “imagine all of the 9’s present at once, but had difficulty seeing 0.\(\bar{9}\) as a number that can be transformed” (Dubinsky et al., in press, p. 12). Though this stage’s applicability to other concepts is under the researchers’ scrutiny, I believe it is necessary to delineate between a process as a totality (without being acted on) and a process as an object (explicitly acted on). For example, observed from my own data, students were able to view a function as an entity, but did not show evidence of acting on the function. Previously, under the old stages, I did not know how to classify this situation; it was more than a process, but it was not quite an object. TOT is an appropriate assessment of this data.

Finally, a schema is a collection of actions, objects, processes, and other schemas (Dubinsky and Mcdonald, 2002) that represents the “totality of knowledge” (Asiala et al., 1996) that is associated with a particular mathematical topic. Thus, schema is not considered part of the hierarchy of action, processes, (totalities), and objects. Schema is not the focus of the current research, so the review of the literature is limited to the two aforementioned studies. However, Section 2.3 presents examples of schemas for point-wise and uniform compositions.

This section has shown that the definition of action has evolved since it was first established. An action conception requires a student to explicitly perform a mathematical task that transforms objects. Holding a process conception is procedurally the same as having an action conception, except that the actions are entirely imagined. External stimuli are also not needed to trigger students to transform an object. Two indications of a process conception include
the ability to reverse the action and the ability to combine the concept with other previously established processes.

Though the totality stage is not firmly established in the literature, evidence from my research shows that this stage is necessary to describe some students’ understanding of mathematical concepts. In this stage, students have the understanding that a particular mathematical concept is a process and also an entity, but has not yet acted on it. Once students realize that the entities can be acted on, they have entered the object stage. They must explicitly act on the totality (as opposed to the process in the original definition) in order for that totality to be fully considered an object. As such, there are two levels to this phase. Though schema comes last, it is not a separate conception since it is made up of the previous stages. Therefore, the object conception is currently the highest stage of APOS theory.

2.2 Theory Applied to Functions

To demonstrate the variance in using APOS to assess students’ function conceptions, I organized definitions of an action conception posed by the creators of the theory and the definitions created by others who have interpreted and used the theory (Table II). Though the definitions share elements of an action conception of function:

- tied to algebraic expression,
- explicit act of substituting numbers for one variable and performing a series of procedural techniques to get another number,
- input/output strongly restricted to integers,
• explicitly solving one variable for another,

there is one major difference. Recall that in the general definition of an action conception, it is a requirement that the student explicitly carries out the steps of the transformation (p. 8). Thus for an action conception of function, the student must explicitly treat functions as a step-by-step procedure, either verbally or written. However, in Table II, the first and last interpretations of the general definition by other researchers show they evaluated students as action if this procedure was imagined. This is in direct contrast with the actual definition of action and will result in different assessments of the same student data. Part of this is due to the shifting nature of a young developing theory, and part of this may be attributed to misinterpretation of theory. Either way, Table II shows that on the path from the theory to raw data, the interpretation of the theory can affect the final assessment of student data. A common APOS data analysis tool can greatly aid assignment of students into the appropriate categories. Furthermore, the data analysis tool would be useful to provide updated definitions of APOS conceptions. Such a tool is presented in Chapter 6.

Recall that a process conception begins procedurally as an action except that the transformation is completely mental; generality is acceptable in a process, but not in an action. As shown in ??, the definitions are fairly representative of the original definitions posited by the creators. The first and second definitions provided by (Breidenbach et al., 1992) explicitly state that results of the transformation must be objects. However, since these output values are not acted on, they are totalities and not objects.
### TABLE II: DEFINITIONS OF ACTION CONCEPTION OF FUNCTION

<table>
<thead>
<tr>
<th>Creators of Theory</th>
<th>Others</th>
</tr>
</thead>
</table>
| • “act of substituting numbers for variables [one evaluation at a time in an explicit algebraic expression] and calculating to get a number, 
  ...[but] explicit mention of beginning or of resulting objects was missing” |
  (Breidenbach et al., 1992, p. 252)                                                 |
| • “[input, output, transformation were all] present, but the procedure was tied to an expression or equation, or if the input or output objects were strongly restricted, say to integers” |
  (Breidenbach et al., 1992, p. 252)                                                 |
| • “explicitly making a calculation [for solving one variable for another] and insisting on having a specific expression” |
  (Dubinsky and Harel, 1992, p. 93)                                                  |
| • “each individual computation must be explicitly performed or imagined”             |
  (Oehrtman et al., 2008, p. 157)                                                    |
| • “input and output are not conceived except as a result of values considered one at a time, so the student cannot reason about a function acting on entire intervals.” |
  (Oehrtman et al., 2008, p. 157)                                                    |
| • “[view] defining formula [of a function] as a procedure for finding an answer for a specific value of $x$; view the formula as a set of directions ...to get the answer” |
  (Oehrtman et al., 2008, p. 158)                                                    |
| • “view functions only in terms of symbolic manipulations and procedural techniques disassociated from an underlying interpretation of function as a more general mapping of a set of input values to a set of output values” |
  (Carlson et al., 2010, p. 115)                                                     |
| • “confined to seeing the defining formula as a computational procedure [or a set of instructions] for finding a single answer for a specific value of $x$” |
  (Carlson et al., 2010, p. 116)                                                     |
| • “imagine each individual operation for an algebraically defined function”         |
  (Carlson et al., 2010, p. 116)                                                     |
The requirements of a process conception of function are explicit mentions of input, output, and transformation. Compared to a student with an action conception of a function, a student with a process conception does not need to explicitly evaluate a function at a number, but instead sees a function as a transformation of any input value, \( n \), to its corresponding output value represented by \( f(n) \). For example, (Breidenbach et al., 1992) classified this student answer, “A function is an operation that accepts a given value and returns a corresponding value,” as a process conception because this answer is general (not restricted to an algebraic expression or equation) and includes the input, output and transformation. However, (Oehrtman et al., 2008) added that there must be a notion of the impact of input values on output values present. That is, it is not enough just to produce outputs with inputs, but to acknowledge the covarying nature of these values. This aligns with the notion of coordinating two processes—mentally attending to the possible input values while also running through the resulting output values—to form a new process, a function.

Several important aspects of a function that were not detailed in the literature on APOS were incorporated into the process conception for this study. For example, the acknowledgement that the transformation maps each input value to a unique output value was included in the process definition of function. Also added is students’ attention to the domain of a function. Students in this study recognized that expressions were not necessarily functions unless they were restricted to certain domains. The last contribution to the process conception of function is an extension of the (Dubinsky and Harel, 1992) statement that ordered pairs are a bellwether for a process understanding. Acknowledgment of other representations of functions
TABLE III: DEFINITIONS OF PROCESS CONCEPTION OF FUNCTION

<table>
<thead>
<tr>
<th>Creators of Theory</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>• “completely mental and so we are not restricting ourselves to computable function in the mathematical sense.” (Breidenbach et al., 1992, p. 251)</td>
<td>• “freed from having to imagine each individual operation for an algebraically defined function” (Oehrtman et al., 2008, p. 158)</td>
</tr>
<tr>
<td>• “able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done” (Breidenbach et al., 1992, p. 251)</td>
<td>• “student can imagine a set of input values that are mapped to a set of output values by the defining expression for ( f )” (Oehrtman et al., 2008, p. 158)</td>
</tr>
<tr>
<td>• “talk about [solving one variable for another] without actually obtaining the expression is probably displaying at least the beginning of a process conception of function” (Dubinsky and Harel, 1992, p. 93)</td>
<td>• “conceive of the entire process as happening to all values at once, and is able to conceptually run through a continuum of input values while attending to the resulting impact on output values” (Oehrtman et al., 2008, p. 158)</td>
</tr>
<tr>
<td>• “set of ordered pairs is a bellwether…for detecting the presence and strength of a process conception” (Dubinsky and Harel, 1992, p. 93)</td>
<td>• “imagine the <em>entire process</em> without having to perform each action” (Oehrtman et al., 2008, p. 159)</td>
</tr>
<tr>
<td>• “subject … think[s] of a function as receiving one or more inputs, or values of independent variables, performing one or more operations on the inputs and returning the results as outputs, or values of dependent variables.” (Asiala et al., 1996, p. 7)</td>
<td>• “A function is a transformation of entire spaces.” (Carlson et al., 2010, p. 116)</td>
</tr>
<tr>
<td>• “sees formula as a means of mapping any input value of the function represented by ( x ) to an output value represented by ( f(x) )” (Carlson et al., 2010, p. 116)</td>
<td>• “when asked to evaluate ( f(3) ) the student thinks about the evaluation as the enactment of a process that takes 3 as an input and produces an output value that is referenced as ( f(3) )” (Carlson et al., 2010, p. 116)</td>
</tr>
</tbody>
</table>
such as graphs, by name (e.g. “parabola”), ordered pairs, or in a table format, were also included as bellwethers. Details are provided in Chapter 6.

A process conception can be observed in many ways, and not all students will show evidence of each one. Therefore, students at the same stage may have very different degrees of understandings of a given concept. These differences, or strength of conceptions, may be useful in explaining why students within the same conception may be displaying different levels of understanding.

Totality and Object conceptions of function are fairly straightforward. Conceptions are recognized as totalities if the result is acknowledged, but there is no action or process applied to the totality. Once a learner recognizes this, s/he has entered the object phase. Explicitly acting on the totality means the student has a full object conception of function. An example of performing an action or process on a totality can be demonstrated through function composition. This is discussed in the next section.

2.3 Theory Applied to Function Composition

Figure 1 is an example of how APOS would be used to show the composition of two functions \( f(x) = x^2 \) and \( g(x) = x + 1 \) at \( x = 2 \). It shows the schema for point-wise composition, i.e., the collection of conceptions students need to perform this operation. If a student traverses from 2 to 9 along the right-hand side of the figure, according to the theory, s/he would have a process conception of function conception. On the contrary, if s/he traverses down the left-hand side, s/he would have an action conception of function composition. Distinguishing how students obtain 9 from 2 is revealed through discussion with the student.
Figure 1: Illustrating APOS conceptions applied to point-wise composition of \( f(x) = x^2 \) and \( g(x) = x + 1 \) at \( x = 2 \)

\[
\begin{align*}
2 & \quad \text{2 is a totality} \\
\downarrow & \\
g(2) & \quad \text{2 is an object} \\
\end{align*}
\]

\[
\begin{align*}
g(x) & \quad \text{is an action} & \text{explicitly evaluates} & \text{imagines evaluation} & g(x) & \quad \text{is a process} \\
\end{align*}
\]

\[
\begin{align*}
g(2) & \quad \text{is a totality} & g(2) = 3 & \quad \text{3 is a totality} \\
\downarrow & \\
f(3) & \quad \text{3 is an object} \\
\end{align*}
\]

\[
\begin{align*}
f(x) & \quad \text{is an action} & \text{explicitly evaluates} & \text{imagines evaluation} & f(x) & \quad \text{is a process} \\
\end{align*}
\]

\[
\begin{align*}
f(3) & \quad \text{is a totality} & f(3) = 9 & \quad \text{9 is a totality} \\
\end{align*}
\]

Notice this evaluation requires at least an object conception of numbers (because numbers are acted on) and at least a process conception of functions (because there must be coordination between two input-output function processes). Thus, having a process conception of function composition is evidence of a process conception of function (Asiala et al., 1996).

Notice that in the composition of Figure 1, there is no reversal or explicit display of coordinating two input-output processes. These can be determined by discussing student thinking with the student. Investigating if students can reverse a composition will reveal the strength of the process conception of function or composition.

The schema for a uniform function composition is slightly different from a point-wise composition. Using the same functions from above, Figure 2 shows what conceptions are needed
to perform this composition. In this operation, students must view a function as an action or process, totality and object. Therefore, if students can perform this composition, it is evident that they have an object conception of function.

2.4 Contributions to APOS

In studies that investigated student cognitive processes using APOS, researchers categorized students as having action, process, or object conceptions of function by using the operational definitions of the conceptions. During data analysis, raw student interview data were fit to these definitions at the discretion of the researchers. As a result, student function conception classification can vary from study to study. This was not due to negligence on the part of the researchers, but rather, to the fact that applying an abstract theory to concrete research data, i.e., student behavior, is very complex. The road that connects the cognitive conceptions of
action, process, and object to actual student behavior is confounded by interpretation of the theory, and interpretation of student behavior. Recent studies (Carlson et al., 2010; Oehrtman et al., 2008) attempt to mitigate this problem by breaking the conceptions into specific function topics such as domain, inverses, etc., and providing operational definitions for them. Some definitions are so specific that they detail raw student interview observations. This is the direction in which this study seeks to take APOS.

To facilitate the use of this theory and to minimize mismatches, a strong chain of reasoning between actual student work, researchers’ interpretation of student work, definitions of action, process, object, and schema for specific topics, and general operational definitions should be established. One outcome of this study is to strengthen the study with these contributions:

- **Modern operational definitions.** The theory undergoes refinement with each use, so the definitions have shifted from what they were originally. An evaluation of the literature in Section 2.1 provided updated operational definitions to prevent researchers from using obsolete definitions.

- **Transition stages.** Students from my study sometimes displayed evidence they were beyond a conception, but not quite at the next stage. Until recently (Dubinsky et al., in press), there were no transition stages in APOS, and the transition levels that do exist for APOS are not for the concept of function.

- **Topic-specific operational definitions.** The original definitions were very general and did not include definitions for specific topics such as function composition, domain or range.
Researchers have made some progress in this area (Carlson et al., 2010; Oehrtman et al., 2008), but it is not complete.

- **Strength levels for each stage of conception.** As shown in the previous sections, definitions have several criteria. Research does not address how to assess students if they do not exhibit all the criteria required to possess a specific conception.

- **Include raw data.** Matching raw student responses to all definitions would give other researchers an idea of how to code their own data the way the creators of the theory intended.

The first three were addressed previously in this chapter where the fourth point was also briefly motivated, but full details on this point and the last one will be provided in Chapter 6. Addressing these areas would facilitate researchers in using this theory, but more importantly it would increase cohesion between theory and raw data, and also between researchers using the same theory, thus increasing the validity and reliability of the theory.

This research aggregates the general operational definitions posed by the APOS developers with the purpose of presenting modern definitions of action, process, object, schema. These definitions were used to check compatibility with topic-specific definitions found in the literature. When topic-specific definitions were unavailable, they were created directly from the modern definitions of the theory. Then raw data from my study was used as exemplars of these definitions (Chapter 6). These exemplars will facilitate coding for future researchers who wish to use APOS, and will also move towards reducing the gap between the intended theory and the enacted theory.
CHAPTER 3

LITERATURE REVIEW

Functions are foundational building blocks of mathematics. The importance of functions is well known within the mathematics community and documented and acknowledged among mathematics education researchers (Dubinsky, 1991; Dubinsky and Harel, 1992; Tall, 1991; Thompson, 1994). The concept has been widely explored both theoretically (Eisenberg, 1991; Harel and Kaput, 1991) and empirically (Ayers et al., 1988; Carlson et al., 2010; Cottrill, 1999; Sfard, 1992). Although more recent research on undergraduate mathematics education has moved away from functions and towards upper level topics such as Linear Algebra and Mathematical Proof, student difficulty with function has not disappeared; as indicated in the title of the (Gooya and Javadi, 2011) paper, “University Students’ Understanding of Function is Still a Problem!”

The following sections will discuss how previous studies have used APOS theory, studied evaluation of functions, and studied function composition.

3.1 APOS Theory

This section continues the discussion from Chapter 2 on APOS theory. In particular, studies that have used APOS theory to design curricula, assessment tools, or research experiments, are described in this section. Most of the literature has been mentioned in the previous chapter; three seminal pieces are discussed here.
(Breidenbach et al., 1992) uses the theoretical perspective of APOS to design a computer-based instructional treatment used to teach elementary discrete mathematics to pre-service teachers. They then used the theory to examine the effect of the instructional treatment on the students' abilities to construct a process conception of function. Researchers gathered information about student understanding of functions through students’ written answers on a questionnaire. The answers were classified by the researchers as prefunction\(^1\), action, or process understandings. The students were given this written instrument twice: once before the instructional treatment and once afterwards. Results from data collected suggest that computer-based instruction improve students' understanding of functions as processes. At the culmination of the study, the researchers refined their definitions of action and process as a result of analysis of interview data.

There is a delicate balance between objects and processes because mathematical concepts like function, present themselves in different contexts, requiring shifts in how we think about them. For example, a function \( f(x) = x^2 + 3 \) can be a process that describes a relationship between two objects \((x \text{ and } f(x))\), but it can also be a solution \((y = x^2 + 3)\) to a differential equation \((y'' - 2 = 0)\). In the latter example, \( y \) is viewed as an object and is acted on through integration. In an even simpler example, take the rational number \( \frac{2}{5} \); it can be an object (the rational number) or a process (dividing 2 by 5).

\(^1\)Students in this conception category have no understanding of function.
(Sfard, 1992) addresses this delicate balance between object and process by discussing them not as two separate things, but instead as different facets of the same thing. (Zandieh, 2000) builds process-object pairs from Sfard’s notion to analyze student understanding of the derivative. In her research, she uses three process-object layers to describe the concept of a derivative, and uses this framework to analyze students’ understandings and to track the development of their understandings.

Though Zandieh’s framework is used for the concept of derivative, it may be applied to other concepts such as function composition, integrals, and differential equations. For instance, (Zandieh, 2000) uses two process-object pairs to describe functions. The first pair, or layer, is “A process of taking an input of a function to its output paired with an object that is the value of the function at a point" (p. 110). As a first layer example, take the function $f(x) = x^2$ for $f : \mathbb{R} \to \mathbb{R}$. The function $f$, is thought of as a process acting on the object 3. The result of this process is the object $f(3) = 9$.

The second process-object pair is “A process of covariation, i.e. imagining ‘running through’ a continuum of domain values while noting each corresponding range value, paired with an object that is the function itself, the set of ordered pairs" (p. 110). The second layer bares a heavier cognitive load of abstracting the idea of function. Understanding the process of function in the second layer requires understanding the effect of acting on all the elements of the domain without actually doing the acting. The resulting object is the set of all ordered pairs.
Process and object conceptions of function at both stages are important for student understanding of functions because students need to be able to evaluate functions at points, and also to understand that a function can be an abstract representation of all inputs and their outputs.

### 3.2 Function Composition

(Ayers et al., 1988) conducted an experiment on post-secondary students to understand their acquisition of the concept of function composition. They studied the effect of computer instantiations on the induction of mental processes (Piaget’s reflective abstraction) when constructing mathematical concepts such as function composition. The teaching intervention used UNIX to teach function composition. Two experimental groups received this treatment and the control group received traditional teaching. Students were given a pretest and posttest to measure whether or not reflective abstraction was enacted. The study found that computer experiences positively impacted students’ ability to understand function composition through enactment of reflective abstraction.

(Cottrill, 1999) studied the relationship between function composition and the chain rule. His work is an extension of work done by (Clark et al., 1997), which used APOS theory to analyze the mental constructions of students applying the chain rule. Cottrill’s study was conducted as a correlational study and also as a multiple case study. Although the quantitative results found only a weak correlation between student performance of the chain rule and their performance of function composition, qualitative results show the importance of understanding function composition in understanding the chain rule.
The qualitative part of the study sought to describe student understanding of the chain rule concept. In his multiple case study, Cottrill used the APOS theory with a focus on the Schema notion to analyze how students understand the chain rule. Despite the lack of correlation in the quantitative part of the study, Cottrill found evidence supporting the notion that the understanding of function composition of functions is key to understanding the chain rule.

3.3 $f(x + h)$

Similar to computation of $f(x + h)$, (Carlson, 1998) briefly documented difficulty with computing $f(x + a)$ for a quadratic expression as part of a larger cross-sectional study. In this study, students from three different levels of study, college algebra, second-semester calculus, and early graduate, were given a 25 item questionnaire. One of the questions asked to compute $f(x + a)$ given $f(x) = 3x^2 + 2x - 4$. The study describes the students from the college algebra group, which also had the lowest mean score on this question, as describing this computation as a substitution of $x$ with $x + a$ (action conception), while the students in the other two groups described it as an evaluation of the function at $x + a$, or that $x + a$ is the input of the function (process conception).

While the most common error that surfaced in Carlson’s study was adding $a$ to the expression, that is, $f(x + a) = f(x) + a$, other more common errors were found in my study. Research from my study seeks to further Carlson’s study by describing the additional errors and explaining what these errors reveal about student cognitive processes while setting up the DQ’s.

Unlike (Carlson, 1998), the current study looks at computing $f(x + h)$ from a function composition perspective; it explores the possibility of using the concept of function composition
as a learning tool to evaluate $f(x + h)$. The few studies on composition focus on the topic itself, its relation to chain rule, or its place in secondary and post-secondary curricula (Ayers et al., 1988; Cottrill, 1999; Horvath, 2011). At the time of this study, no studies view composition in relation to evaluating $f(x + h)$, even though it is mathematically recognized as a composition (Horvath, 2011).
CHAPTER 4

PHASE 1: QUANTITATIVE RESEARCH

4.1 Research Design

The purpose of this phase of the study is to answer Research Question 1. In this section, I will discuss the methods used to identify and categorize the errors that calculus students make when evaluating the limit definition of the derivative. I will include the research method used, participant recruitment, and a description of quiz development. I will also explain the data collection procedures and data analysis procedures. Findings will be presented in the Section 4.2.

4.1.1 Research Method

One of the goals of the study is to describe the trends in the errors students from traditional lecture style calculus courses make when setting up the limit definitions of the derivative. Since survey research “describe[s] trends about responses to questions and to test research questions or hypotheses” (Creswell, 2008, p. 388) it is an appropriate methodology to use in this phase of the study.

Students’ answers on quizzes were examined and analyzed for patterns. T-tests, and ANOVAs were used to determine if there were significant differences between groups and within groups, respectively. Pearson product-moment correlations were used to determine whether there were
significant relationships between quiz scores on routine composition problems and on evaluating the DQ’s.

4.1.2 Participant Recruitment and Description of Participants

In Fall 2010, students from a large Calculus I lecture (Lecture A) were invited to participate in this phase of the research. Recruitment announcements were made during the lecture portion of the course and also during the discussion sections. To strengthen the study, another large Calculus I lecture (Lecture B) was invited to perform a replication study. There were a total of 83 students invited from Lecture A and 63 from Lecture B. Both classes were treated the same way in recruitment.

Lecture A and Lecture B were taught in the traditional lecture style manner for 3 hours a week. Lectures were accompanied by discussion sections that met twice a week. Though midterm exams were created by the respective lecturers, both classes were given a common final. The same textbook Single Variable Calculus: Early Transcendentals (Rogawski, 2008), was used for both classes.

Participants agreed to allow their work on three quizzes to be scanned and analyzed. The number of students that consented for each quiz is presented in Table IV. Although Lecture A had more enrolled students and a smaller percentage of students who participated, the numbers of students that consented in each lecture for every quiz (58 and 49, 52 and 49, 52 and 42, respectively) are relatively close to each other.

It is important to note that the numbers in Table IV do not always represent the same group of students. For example, it is not true that the 52 students who consented to Quiz 2 are a
TABLE IV: NUMBER OF STUDENTS WHO CONSENTED BY LECTURE AND QUIZ

<table>
<thead>
<tr>
<th></th>
<th>Lecture</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A</td>
<td>B</td>
<td>Total</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Invited</td>
<td>83</td>
<td>63</td>
<td>146</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quiz 1</td>
<td>58</td>
<td>49</td>
<td>107</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quiz 2</td>
<td>52</td>
<td>49</td>
<td>101</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quiz 3</td>
<td>52</td>
<td>42</td>
<td>94</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

subset of the 58 from Quiz 1. From Quiz 1 to Quiz 2, there were student absences and attrition, but new students also decided to participate. Similarly, the 52 students from Quiz 2 and 52 from Quiz 3 in Lecture A, are not the same group of students. As such, the total number of participants exceeds 58 and 49 for each lecture.

Overall, 66 students from Lecture A, and 57 from Lecture B participated in this phase of the study. Their majors and year of study are listed in Table V and Table VI, respectively. Although most of the majors are in the sciences, there are a few Liberal Arts majors. Table VI shows that most of the participants were freshmen at the time of the study, and it also shows that the numbers of freshman and sophomore participants were about the same in each lecture. Lecture A also had a graduate student participant.

\[1\] Data on gender could not be confirmed.
### TABLE V: PARTICIPANTS’ MAJORS BY LECTURE

<table>
<thead>
<tr>
<th>Major</th>
<th>Lecture A</th>
<th>Lecture B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Architecture</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Biochemistry</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Bioengineering</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Biological Sciences</td>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>Chemistry</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Civil Engineering</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Computer Engineering</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Computer Science</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Earth &amp; Environmental Sciences</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Economics</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Electrical Engineering</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Engineering - Undeclared</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>English</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Industrial Engineering</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Kinesiology</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Liberal Arts - Undeclared</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td>Mathematical Computer Science</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Mathematics</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Mechanical Engineering</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Psychology</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Public Health</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Teaching of History</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Unknown</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>66</strong></td>
<td><strong>57</strong></td>
</tr>
</tbody>
</table>
TABLE VI: PARTICIPANTS’ YEARS OF STUDY BY LECTURE

<table>
<thead>
<tr>
<th></th>
<th>Lecture A</th>
<th>Lecture B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freshman</td>
<td>20</td>
<td>28</td>
</tr>
<tr>
<td>Sophomore</td>
<td>19</td>
<td>17</td>
</tr>
<tr>
<td>Junior</td>
<td>13</td>
<td>4</td>
</tr>
<tr>
<td>Senior</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>Graduate</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Unknown</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>66</td>
<td>57</td>
</tr>
</tbody>
</table>

4.1.3 Quiz Development

Another investigator and I created four routine questions that reflected the usual questions given on a weekly calculus quiz. The four questions were dispersed over three quizzes (see Appendix A). These questions were then shown to and pre-approved by the instructors and the teaching assistants of both lectures. Students answered these questions in quizzes during the discussion section as part of the regular calculus course. Two of the questions asked students to set up the difference quotients for various functions. Another question asked students to evaluate functions at $2 + h$ and the last question was a function composition question. The main purpose of these questions was to identify errors and describe trends in the questions, errors, lectures, and function type. Details of the purpose of each question can be found in the following paragraphs.
Each of the four questions had three parts, each part representing a different kind of function: monomial, rational function, trigonometric function. The purpose of these different types of functions is to answer the second part of Research Question 1): Are the errors committed independent of the function type? Or does the type of error change for different types of functions?

Questions from Quiz 1 (see Appendix A.1) were not a part of the original research. While hypothesizing about the errors of the difference quotient in the limit definition of the derivative function (abbreviated as $DQ_h$):

$$f(x + h) - f(x)$$

we became curious about the students’ performance on setting up the difference quotient in the limit definition of the derivative at the point $a$ (abbreviated as $DQ_a$):

$$\frac{f(x) - f(a)}{x - a}.$$

We were curious about how students’ answers from these two difference quotient questions compared with each other, but the main purpose of the first question on Quiz 1 was to test this hypothesis: students have more difficulty with the set up of $DQ_h$ than with $DQ_a$.

The second question on Quiz 1 was developed to determine if student difficulty with $f(x + h)$ was a result of a weak understanding in the role of $x$ as an abstract variable. We replaced $x$ by a concrete number and asked students to evaluate $f(2 + h)$ for certain given functions.

The purpose of the third question is to record and categorize errors that students make when setting up $DQ_h$ (see Appendix A.2). The scores on this question will be correlated with those
of the last question. The fourth and final question asks students to answer routine function composition problems that are similar to $f(x + h)$ (Appendix A.3). I wanted to know what kind of relationships exist between performance on the composition problem compared with the $f(x + h)$ evaluation required in the DQ$_h$ problem. Of particular interest was the group who failed to evaluate $f(x + h)$ correctly, but can compose the functions. Evidence of this would imply that students do not realize that $f(x + h)$ can be thought of as a composition of two functions.

Select student work from each question will be presented to participants from Phase 2 of the study. Each participant from Phase 2 was asked to comment on erroneous student work, as well as their own student work.

4.1.4 Data Collection Procedures

The four questions were included in weekly quizzes, Quiz 1, 2, and 3, respectively, and administered to all students during Weeks 4, 5 and 6 of the 15 week course, respectively, as a part of the weekly quiz.\textsuperscript{1} The quiz questions developed in this phase did not comprise the entire quiz, but instead accompanied additional quiz questions created by each respective teaching assistant. The timing of the questions aligned with the weeks that students were to be quizzed on the corresponding topic. Due to unforeseen circumstances, quiz distributions differed in the two lectures. This is detailed in the next few paragraphs.

\textsuperscript{1}The enumeration of the quizzes are given in reference to the study and not as they were called in the course.
Quiz 1 had two questions from the study, and each question was worth 3 points, so the maximum score a student could have is 6 points. Quiz 2 only had one question from the study so the maximum possible score was 3 points. In Lecture A, students took Quiz 1 and Quiz 2 in separate weeks, as planned. However, in Lecture B, questions from Quiz 1 and 2 were combined and presented on one quiz.

Although distribution of the instruments from which the data were gathered were not identical in Lectures A and B, an independent samples t-test was employed and showed no significant difference in mean scores of the questions from the two quizzes of Lectures A and B. The next two paragraphs explain the details of the statistical analysis.

An independent samples t-test compared the scores of students who answered the three questions from both Quiz 1 and Quiz 2 in Lecture A, to the scores of students who answered the same questions on a combined quiz in Lecture B. No participants from Lecture B were excluded in this test because they all answered the three questions. However, in Lecture A, if data for a particular student were incomplete for any one of the quizzes (the student did not take the quiz or did not consent to have quizzes analyzed), then that student’s data was not included in this analysis. This is to ensure that the sample sizes from both lectures are based on the number of students who answered all three questions. The sample sizes are included in Table VII.

Table VII also shows the mean scores of the quizzes in each lecture. There is no statistical difference ($\alpha = .05$) in mean scores between Lecture A ($M = 3.89, SD = 2.11$) and Lecture B ($M = 4.00, SD = 1.99$) for Quiz 1, $t_1(94) = -.26$, or for Quiz 2, $t_2(94) = .36$. This indicates
TABLE VII: MEAN SCORES BETWEEN GROUPS

<table>
<thead>
<tr>
<th>Lecture</th>
<th>A</th>
<th>B</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>47</td>
<td>49</td>
<td>–</td>
</tr>
<tr>
<td>Quiz 1</td>
<td>$M$</td>
<td>3.89a</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>$SD$</td>
<td>2.11</td>
<td>1.99</td>
</tr>
<tr>
<td>Quiz 2</td>
<td>$M$</td>
<td>2.47</td>
<td>2.39</td>
</tr>
<tr>
<td></td>
<td>$SD$</td>
<td>1.06</td>
<td>1.13</td>
</tr>
</tbody>
</table>

$^a$ All statistics are rounded to the nearest hundredth.

data that answering the three questions on two separate quizzes as in Lecture A, or all together on one quiz as in Lecture B, had no significant statistical impact on their scores.

4.1.5 Data Analysis Procedures

The answers were analyzed with the purposes of categorizing the errors that students made in each of the four questions and of describing the trends in the errors, questions, function type, and lecture. This section details the analysis process of the data from the quizzes.

After the data were collected from the students, the quizzes were scanned and graded. The three parts of each of the four questions were graded on a binary system:

$$grade(\text{answer}) = \begin{cases} 
1 & \text{if } \text{answer} \text{ is completely correct} \\
0 & \text{otherwise.}
\end{cases}$$

Each part was graded with leniency; wrong answers were ignored if the correct answer was present. For each question, the highest possible score was 3.
To answer the first part of Research Question 1, error categories were identified. Initially, each answer was either placed into the “correct answer” category, its own error category if that error type had not been identified, or in a pre-existing error category. The error categories were then refined by organizing errors of which there were 3 or less total occurrences, into one general category called “Other”. To answer the second part of Research Question 1 and also to analyze trends between errors, lectures, function types, and quiz questions, data were put into the quantitative analysis program PASW (SPSS).

4.2 Findings

I will report the findings from this phase of the study in this section, and draw conclusions from these findings. I will begin by discussing findings from each question and then show findings from data across questions. Before I do this, I will report the mean scores of each question for both lectures. Independent samples t-tests were employed to compare mean scores of each question in Lecture A and Lecture B. Table VIII shows that there are no statistical differences ($\alpha = .05$) in mean scores of Lecture A and Lecture B, for any of the four questions: $t_1(105) = -.24$, $t_2(105) = -.29$, $t_3(99) = .34$, and $t_4(91) = -1.37$, respectively. Therefore, for the purposes of this study, Lecture A and Lecture B are good choices for a replication study.

4.2.1 Question 1: Evaluate DQ_a (Quiz 1, Appendix A.1)

Figure 3 is an example of student work that was considered completely correct. This student received one point for each function type (monomial, rational function and trigonometric function) for which s/he correctly set up DQ_a, totaling 3 points on this question. In contrast, Figure 4 is an example of student work that received a score of 0; for each function type,
<table>
<thead>
<tr>
<th>Question</th>
<th>Lecture</th>
<th>A</th>
<th>B</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>N</td>
<td>58</td>
<td>49</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>M</td>
<td>1.38</td>
<td>1.45</td>
<td>.81</td>
</tr>
<tr>
<td></td>
<td>SD</td>
<td>1.49</td>
<td>1.5</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>N</td>
<td>58</td>
<td>49</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>M</td>
<td>2.50</td>
<td>2.55</td>
<td>.78</td>
</tr>
<tr>
<td></td>
<td>SD</td>
<td>.96</td>
<td>.87</td>
<td>–</td>
</tr>
<tr>
<td>3</td>
<td>N</td>
<td>52</td>
<td>49</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>M</td>
<td>2.46</td>
<td>2.39</td>
<td>.74</td>
</tr>
<tr>
<td></td>
<td>SD</td>
<td>1.08</td>
<td>1.13</td>
<td>–</td>
</tr>
<tr>
<td>4</td>
<td>N</td>
<td>52</td>
<td>42</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>M&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1.75</td>
<td>1.26</td>
<td>.89</td>
</tr>
<tr>
<td></td>
<td>SD</td>
<td>.56</td>
<td>.58</td>
<td>–</td>
</tr>
</tbody>
</table>

<sup>a</sup> All questions have a possible total score of 3 except the last one which has a possible total of 2 due to a typo. Explanation in discussion of Question 4.
Figure 3: Sample of Correct Student Work for Question 1

1. Set up the difference quotient,

\[ \frac{f(x) - f(a)}{x - a}, \]

for each of the following functions:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin x \)

The student failed to correctly set up DQ\(_a\).\(^1\) This sample is taken from the pool of students who followed the directions as stated; there was a subset of students who did not use DQ\(_a\) and instead used DQ\(_h\).\(^2\)

The main purpose of Question 1 was to compare the results with results to Question 3, which asked to set up DQ\(_h\). This comparison will be discussed in a later subsection that reports findings on the relationship between all questions. The current subsection will report the findings of Research Question 1 as it pertains to this quiz question. That is, to identify errors and also to

\(^1\) Notice that in Figure 4, the question is labelled as question 2. The enumeration of the questions are given in reference to the study and not as they were called in the course.

\(^2\) Figure 4 shows the student wrote DQ\(_h\) next to DQ\(_a\) in this example, but did not proceed to use it.
2. Setup the difference quotient:

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{f(x) - f(a)}{x - a}
\]

for each of the following functions:

(a) \( f(x) = x^2 \)
\[
\frac{f(x)^2 - f(a)}{x^2 - a}
\]

(b) \( f(x) = \frac{1}{x} \)
\[
\frac{f(\frac{1}{x}) - f(a)}{\frac{1}{x} - a}
\]

(c) \( f(x) = \sin(x) \)
\[
\frac{f(\sin(x)) - f(a)}{\sin x - a}
\]
investigate if there was a difference in ability to set up \( DQ_a \) for different function types. Analysis of data shows that there is no difference in mean scores when setting up the \( DQ_a \) for different function types. To support this, converging evidence from multiple forms of data analysis are presented in the following paragraphs.

The first type of analysis is statistical in nature. A test in equality of means using a one-way within subjects (or repeated measures) ANOVA was conducted to compare the effect of function types of monomial, rational function and trigonometric function, on question score.\(^1\) There was no statistical difference (\( \alpha = .05 \)) in scores for each function type, \( F(2, 210) = 1.34, p = .27 \). This result shows that function type does not have an effect on the ability to set up the \( DQ_a \).

Student scores on the question reflect the same result. Table IX reports the percentages of students who received a score of 0 for any three parts of question 1. For example, 53% of all the participating students from Lecture A scored a 0 for failure to set up \( DQ_a \) for the monomial, \( f(x) = x^2 \). The numbers show that for each type, there is no significant difference in the percentages of students who answered the question incorrectly (53%, 55%, 53%).\(^2\) This suggests that function type does not have an effect on the ability of students to set up \( DQ_a \); that is, the students either correctly set it up for all the function types, or for none. Not only do these similar percentages suggest that function type has no effect on students’ ability to evaluate \( DQ_a \), but the percentages show that in general, it is difficult for the majority of students to

---

\(^1\) Mauchly’s test did not show a violation of sphericity against function type, \( \chi^2(2) = .02, p = 0.99 \).

\(^2\) All percentages are rounded to the nearest whole percent.
evaluate it. Data from Lecture B support both these claims; over half of the students got this question wrong and for each function type, there was no significant difference in the percentages of students who answered the question incorrectly (51%, 53%, 51%).

The breakdown of the scores on this question by lecture in Figure 5 shows the same results. It shows the numbers and percentages of students that received each of the possible scores of 3, 2, 1, 0 on this question. The percentages of students who got a perfect score in each lecture are similar (45% and 47%), as are the percentages who received 0: 52% and 51%, respectively. It is interesting to note that in both lectures, there were a negligible number of students who received a score of 2 or 1. This says that if a student answered the question correctly for one function type, they likely answered the question correctly for the other two function types and likewise for incorrect answers. The results from this figure support the same statement as the statistical analysis and Table IX: the evaluation of DQa for each function type is of similar difficulty for the students.

<table>
<thead>
<tr>
<th>Lecture</th>
<th>$x^2$</th>
<th>$\frac{1}{x}$</th>
<th>$\sin x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>53</td>
<td>55</td>
<td>53</td>
</tr>
<tr>
<td>B</td>
<td>51</td>
<td>53</td>
<td>51</td>
</tr>
</tbody>
</table>
The data tables discussed thus far have been about analyzing student scores for the purpose of investigating whether function type had an effect on the ability to set up $DQ_a$. In order to investigate whether function type has an effect on the type of error committed, I now shift my focus to the errors. Table X shows the percentages of errors made for each function type in each lecture out of the total instances of error in that lecture. For example, in Lecture A, 34% of the errors made in setting up $DQ_a$ were made with the monomial, $f(x) = x^2$. In Lecture A, the percentages of errors made for each function type are distributed fairly evenly between the three types. This is further evidence that all function types pose a similar amount of difficulty for students when setting up $DQ_a$. Results from the replicated study done in Lecture B reflect...
TABLE X: PERCENTAGES OF ERRORS MADE IN EACH LECTURE FOR QUESTION 1

<table>
<thead>
<tr>
<th>Lecture</th>
<th>( f(x) )</th>
<th>( x^2 )</th>
<th>( \frac{1}{x} )</th>
<th>( \sin x )</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>34</td>
<td>34</td>
<td>32</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>33</td>
<td>34</td>
<td>33</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

the same finding; percentages of error are similar across function types and they are also very close to the percentages of Lecture A.

These totals will now be broken down into particular error categories in order to identify the most frequent errors. These error categories will be discussed with the aid of the student work in Figure 4. Starting with part (a), \( f(x) = x^2 \), I will pick out the errors going from top to bottom and left to right. Surprisingly, for \( f(x) \) in the difference quotient, instead of putting \( x^2 \), the student put \( f(x)^2 \). It seems the student is using \( f(x) \) to denote two different things; the student seems to separate the \( f(x) \) in \( \text{DQ}_a \) from the \( f(x) = x^2 \) given in part (a). To clarify the distinction of the two \( f(x) \)'s, let's rename the \( f(x) = x^2 \) as \( g(x) = x^2 \). If we do that, then it seems the student has taken \( f(x) \) [in \( \text{DQ}_a \)] and used it as an input for \( g(x) = x^2 \) to get \( g(f(x)) = f(x)^2 \).

Next, the student did not evaluate \( f(a) \) in the numerator. In the denominator, instead of keeping it as it is in \( \text{DQ}_a \), s/he has replaced \( x - a \) with \( x^2 - a \). Moving to (b) and (c), at first glance, it seems the student has made the same errors as in part (a), but there is a notable
difference between the work in (a), and the work in (b) and (c). If we again reassign \( f(x) = \frac{1}{x} \) to \( g(x) = \frac{1}{x} \), and keep \( f(x) \) to denote the \( f(x) \) in the DQ\(_a\), then it seems the student has done \( f(g(x)) = f(\frac{1}{x}) \). Similarly for (c), \( f(g(x)) = f(\sin(x)) \).

The errors found in this student’s work represent some common errors found in both lectures. These common errors are organized in Table XI along with three additional errors not found in Figure 4.\(^1\) The first addition to the previously identified categories is Error I, “Used DQ\(_b\)”. In this error category, students did not use DQ\(_a\) as instructed.\(^2\) The second additional error category, Error III, “\( f(a) \) as \( a \)”, comes from students failing to evaluate \( f(a) \) in DQ\(_a\) to get \( a^2 \), \( \frac{1}{a} \), or \( \sin a \), and instead replacing it with \( a \) in every instance. The third and last addition is the “Other” category, an aggregate of blank answers, incomprehensible answers, and error categories that had fewer than 3 members. This error category included answers with \( f(a) \) evaluated similar to how \( f(x) \) was evaluated in error categories \( f(g(x)) \) and \( g(f(x)) \) (e.g. \( f(a)^2 \), \( f(\frac{1}{a})^2 \), or \( f(\sin a) \)), answers with DQ\(_a\) as input into the given functions, and one solution presented as a derivative using the quotient rule.

The converging pieces of evidence presented have shown that Lecture A and Lecture B are similar with respect to mean scores and difficulty with problems. To gain a better understanding of the errors, we now turn our attention to them exclusively, without regard to lecture. Thus far,

\(^1\)As indicated in the incorrect work presented in Figure 4, some students made more than one error in the individual parts of the question. It is important to note that in Table XI, students’ work possibly fell into multiple error categories.

\(^2\)Discussion of this error category will be in the section on findings from all four quiz questions.
TABLE XI: PERCENTAGE OF ERROR DISTRIBUTION WITHIN EACH FUNCTION TYPE FOR QUESTION 1

<table>
<thead>
<tr>
<th>Error Category</th>
<th>$x^2$</th>
<th>$\frac{1}{x}$</th>
<th>$\sin x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Used DQ$_a$</td>
<td>17</td>
<td>18</td>
<td>16</td>
</tr>
<tr>
<td>II. No $f(a)$ evaluation</td>
<td>37</td>
<td>36</td>
<td>38</td>
</tr>
<tr>
<td>III. $f(a)$ as $a$</td>
<td>6</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>IV. $g(f(x))$$_b$</td>
<td>9</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>V. $f(g(x))$$_c$</td>
<td>10</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>VI. Denominator as $g(x) - a$$_d$</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>VII. Other</td>
<td>10</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td><strong>Total</strong>$^e$</td>
<td><strong>100</strong></td>
<td><strong>100</strong></td>
<td><strong>100</strong></td>
</tr>
</tbody>
</table>

$^a$ includes students who used DQ$_a$ incorrectly or correctly.

$^b$ $f(x)^2$, $\frac{1}{f(x)}$, and $\sin(f(x))$, respectively.

$^c$ $f(x^2)$, $f(\frac{1}{x})$, and $f(\sin x)$, respectively.

$^d$ $x^2 - a$, $\frac{1}{x} - a$, and $\sin(x) - a$, respectively.

$^e$ These totals are accurate for actual percentages and not for reported nearest percent estimates.
we have seen that different function types had no effect on ability to set up DQ\textsubscript{a}. However, when analyzing distribution of errors within individual function types and across all error categories, we begin to see that function type did matter for type of error committed.

We inspect the distribution of the errors within individual function types and across all error categories in Table XI to show which error types occur the most frequently in each function type. That is, all identified error types are considered together in one function type and then again for each of the remaining function types. For example, 37\% of all the errors made with function type \( x^2 \) were in the error category “No \( f(a) \) evaluation”. For all function types, this is the most frequent error made (37\%, 36\%, 38\%). This error and the next error, “\( f(a) \) as \( a \)”, make up almost half of all the errors made for each function type. This table also shows 9\% of the total errors made with the monomial were of the form \( f(x)^2 \), (error type \( g(f(x)) \)), while that error occurred only 1\% with the rational function, and 0\% with the trigonometric function. In addition, we see that 18\% of all errors made with \( \frac{1}{x} \), were of the error type \( f(g(x)) \) (error \( f(\frac{1}{x}) \)); in comparison, only 10\% of all errors made with \( x^2 \), were made in this error category (error \( f(x^2) \)). This suggests that function type affected which errors were made in the set up of DQ\textsubscript{a}. Although this table can be used to come to this conclusion, it is only designed to report on how prominent an error type was for individual function types and not to generalize across function types.

We now turn to analysis to generalize across function types; that is, we will compare the percentages of error across all function types within individual error types. Table XII shows which function type was most prevalent for each error type. For example, 33\% of all the errors
made in the category III, “$f(a)$ as $a$”, were made with the monomial function type, 40% made with the rational function, leaving the rest made with the trigonometric function. In this error category, the function type with the highest percentage of occurrence is with the rational function, $\frac{1}{x}$. For that same function type, in the error category V, $f(g(x))$, 40% of the errors were written as $f\left(\frac{1}{x}\right)$, as opposed to the 23% in the monomial, which was written as $f(x^2)$. The distribution of the errors across function types seems to be fairly similar except for the error type $f(g(x))$. For this error category, 88% of the errors were committed when setting up $DQ_a$ for the monomial $x^2$ (error $f(x)^2$), while 13% gave $\frac{1}{f(x)}$ for the rational function, and no answers were given in the corresponding form for the trigonometric function. In the “Other” error category, students seem to make these errors more often if the function was $\sin x$. Although function types seem to have no effect on the ability to set up $DQ_a$, it does seem to affect which errors were committed.

Although the data from statistical analysis, Table IX, Table X and Figure 5 converged to show that all three function types posed a similar amount of difficulty for students, data from Table XI and Table XII converged to show that the reason for difficulty depended on function type. In particular, the error $f(g(x))$ for $x^2$ was made more frequently than for $\frac{1}{x}$ and $\sin x$, while $g(f(x))$ occurred a lot more frequently than $f(g(x))$ for both rational and trigonometric functions.

4.2.2 Question 2: Evaluate $f(2 + h)$ (Quiz 1, Appendix A.1)

As stated earlier, the main purpose of this question was to compare student answers to how students evaluate $f(x + h)$ in $DQ_h$ in Question 3. This comparison will be discussed later in
TABLE XII: PERCENTAGE OF ERROR DISTRIBUTION ACROSS FUNCTION TYPES FOR QUESTION 1

<table>
<thead>
<tr>
<th>Error Category</th>
<th>$g(x)$</th>
<th>$x^2$</th>
<th>$\frac{1}{x}$</th>
<th>$\sin x$</th>
<th>Total$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Used DQ$_h$$^b$</td>
<td></td>
<td>33</td>
<td>36</td>
<td>31</td>
<td>100</td>
</tr>
<tr>
<td>II. No $f(a)$ evaluation</td>
<td></td>
<td>33</td>
<td>33</td>
<td>33</td>
<td>100</td>
</tr>
<tr>
<td>III. $f(a)$ as $a$</td>
<td></td>
<td>33</td>
<td>40</td>
<td>27</td>
<td>100</td>
</tr>
<tr>
<td>IV. $g(f(x))$$^c$</td>
<td></td>
<td>88</td>
<td>13</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>V. $f(g(x))$$^d$</td>
<td></td>
<td>23</td>
<td>40</td>
<td>37</td>
<td>100</td>
</tr>
<tr>
<td>VI. Denominator as $g(x) - a$$^e$</td>
<td></td>
<td>33</td>
<td>33</td>
<td>33</td>
<td>100</td>
</tr>
<tr>
<td>VII. Other</td>
<td></td>
<td>30</td>
<td>30</td>
<td>41</td>
<td>100</td>
</tr>
</tbody>
</table>

$^a$ These totals are accurate for actual percentages and not for reported nearest percent estimates.

$^b$ includes students who used DQ$_h$ incorrectly or correctly.

$^c$ $f(x)^2$, $\frac{1}{f(x)}$, and $\sin (f(x))$, respectively.

$^d$ $f(x^2)$, $f(\frac{1}{x})$, and $f(\sin x)$, respectively.

$^e$ $x^2 - a$, $\frac{1}{x} - a$, and $\sin (x) - a$, respectively.
the subsection that describes findings on the relationship between all questions. The structure of this subsection will be similar to the structure of the subsection for findings from Question 1; it begins with samples of correct and incorrect student work, and then it describes multiple analyses of the data in order to show converging evidence that function type influences the ability to answer the question correctly (unlike in Question 1) and also that function type influences the type of error made. These analyses include using an ANOVA test, data comparison tables of scores, and tables revealing error categories. Each of these analyses is followed by discussion of the findings of Research Question 1 as it pertains to this question. In other words, errors are identified and the ability to answer this question for different function types is investigated.

Figure 6 is an example of student work that was considered completely correct and received a score of 3. For the function type $x^2$, $(2 + h)^2$ was accepted as a fully correct answer; students were not expected to expand the binomial. In comparison, Figure 7 is an example of student work that received a score of 0. Notice in the latter figure that the student seemed to have used DQ_a to answer the question even though the question did not ask to do so. Many students did this so in grading this, I took the first part of the numerator of the student’s work, $f(2 + h)^2$, as the answer for $f(2 + h)$ and all additional information was ignored and ungraded.

A test in equality of means using a one way within subjects ANOVA was conducted to compare the effect of function types, monomial, rational function and trigonometric function, on question score. Mauchly’s test indicated that the assumption of sphericity had been violated at $\alpha = .05$ ($\chi^2(2) = 10.12, p = .006$), therefore degrees of freedom were corrected using Greenhouse-Geisser estimates of sphericity ($\epsilon = .92$). The results show that there is a significant difference
Figure 6: Sample of Correct Student Work

3. Please evaluate \( f(2 + h) \) for each of the same functions above:

(a) \( f(x) = x^2 \)
\[
  f(2 + h) = (2 + h)^2 = 4 + 4h + h^2.
\]

(b) \( f(x) = \frac{1}{x} \)
\[
  f(2 + h) = \frac{1}{2 + h}
\]

(c) \( f(x) = \sin x \)
\[
  f(2 + h) = \sin (2 + h)
\]

Figure 7: Sample of Incorrect Student Work

2. Evaluate \( f(2 + h) \) for each of the same functions above:

(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

(a) \[
  \frac{f(2 + h)^2 - f(2)}{2h - 2}
\]

(b) \[
  \frac{f(2 + h) - f(2)}{h}
\]

(c) \[
  \frac{f(\sin (2 + h)) - f(\sin 2)}{\sin (2 + h) - \sin 2}
\]
at $\alpha = .05$, in the scores of the three function types, $F(1.83, 192.18) = 7.63$, $p = .001$. Post hoc tests using the Bonferroni correction revealed that there is a significant difference at $\alpha = .05$ between the scores on the function types monomial and trigonometric ($p = .003$).

Students’ 0 scores on the question reflect the same result. Table XIII reports the percentages who received a score of 0 for any three parts of question 2. It is clear from this table that there is a significant difference between the percentage of students who got this question wrong for the monomial and the trigonometric function. For the monomial function, low percentages suggest that most students did not have much difficulty with this question. The highest percentages in Lectures A and B were 24% and 20% wrong for $\sin x$, respectively. In both lectures, the higher percentages for $f(x) = \frac{1}{x}$ and $f(x) = \sin x$ suggest that these functions gave more students difficulty than $f(x) = x^2$ did when evaluating $f(2 + h)$.

Figure 8 shows the breakdown of the scores on this question by lecture. It shows the numbers and percentages of students that received each of the possible scores of 3, 2, 1, 0. The percentages of students who got a perfect score in each lecture are similar (74% and 76%). With such high

\begin{table}[h]
\centering
\caption{Percentage of Students in Each Lecture Who Scored 0 for Question 2}
\begin{tabular}{llll}
\hline
Lecture & $f(x) = x^2$ & $f(x) = \frac{1}{x}$ & $f(x) = \sin x$ \\
\hline
A & 10 & 16 & 24 \\
B & 8 & 16 & 20 \\
\hline
\end{tabular}
\end{table}
percentages of students who got this entire question correct, it is clear that it did not pose a huge problem for the students. The rest of the student scores are distributed over all the possibilities of 2, 1, or 0. This tells us that of the students who made errors, function type seems to affect their performance of evaluating $f(2 + h)$, but it does not tell us with which function type the errors were made.

Thus far, I have focused on the analysis of student scores, for the purpose of investigating whether function type affected the ability to evaluate $f(2 + h)$. I now shift my focus to the errors in order to investigate whether function type affects the type of error committed. Table XIV
shows the percentages of errors made for each function type out of the total instances of error by lecture. For example, 21% of all the errors made in Lecture A were made with the monomial function type, 31% were made with the rational function, and the largest percentage of the errors were made with the trigonometric function. In Lecture A, it is clear that evaluating $f(2 + h)$ for $x^2$ is the least difficult and $\sin x$ is the most difficult; function type seems to have an effect on evaluation of $f(2 + h)$. This supports the previous findings that there is a significant difference between the number of errors made with the monomial and the trigonometric function. Results from the replicated study done in Lecture B reflect the same finding.

Like the analysis done for Question 1, all student work on Question 2 was parsed into categories and data from lectures were combined. Only three major categories were identified in Question 2. Student work from Figure 7 shows two of the three categories. Part (a) shows that the student sees the $f$ in $f(2 + h)$ as different from $f(x) = x^2$. If we rename the latter $f(x)$, we get $g(x) = x^2$, and it seems the student has used $f(2 + h)$ as an input for $g(x)$, i.e., $g(f(2 + h))$ to get $f(2 + h)^2$. In parts (b) and (c), the functions will also be renamed to facilitate discussion.
We get $g(x) = \frac{1}{x}$, and $g(x) = \sin x$, respectively. Unlike part (a), the student seems to be using the $g(2 + h)$ as input for $f(x)$, i.e., $f(g(2 + h))$. In other words, the student has given the answers, $f(\frac{1}{2 + h})$ and $f(\sin (2 + h))$, respectively. Not shown in this figure, another common error was giving the answer $\sin 2 + h$ for the last part of the question. This error category is called “$f(2) + h$” for the purposes of aligning the name of this error type for this question with a similar error type found in the DQ$_h$ question.$^1$ Due to the large numbers of students who left answers blank, a separate and fourth error category was created. Incomprehensible answers were put into the category “Other”. All but two of the errors in this category are unique. The one error that showed up twice was of the form $g((2 + h) + h)$, i.e. for $g(x) = x^2$ this would look like $((2 + h) + h)^2$, and $g(x) = \frac{1}{x}$ looks like $\frac{1}{2 + 2h}$. Although Figure 7 shows additional errors that do not belong to any of these error categories, these errors were not a result of evaluating $f(2 + h)$ and were considered “additional information” and ignored.

These error categories are shown in Table XV along with the percentages of error distribution for each function type. In other words, this table shows the distribution of errors within each function type and across all error categories. For example, all the errors made in evaluating $f(2 + h)$ for $x^2$ fell into three of the five categories: $g(f(2 + h))$, Blank, and Other (33%, 8%, and 58%, respectively). By contrast, evaluation of $f(2 + h)$ for $\sin x$ produced errors in the categories, $f(g(2 + h))$, $f(2) + h$, Blank, and Other (17%, 13%, 42%, and 29%). Note that of the errors in the three major categories identified (the first three in the table), all of the errors

$^1$It is possible that students may have left the parentheses out. Evidence against this is presented in the analysis of DQ$_h$. 
made with $x^2$ fell into the first category, while for the trigonometric function, all of the errors fell into the latter two categories. This tells us that function type affects the errors made in evaluating $f(2+h)$.

We will now organize data to speak on the distribution of errors made within each error category, and across different function types. Table XVI differs from the previous one in that it captures the distribution of a particular error for each function type, rather than displaying the most common error type for a particular function type. For example, 80% of all the errors made in the error type, $g(f(2+h))$, were made with the monomial. All the errors made in the
TABLE XVI: PERCENTAGES OF ERROR DISTRIBUTION ACROSS EACH FUNCTION TYPE FOR QUESTION 2

<table>
<thead>
<tr>
<th>Error Category</th>
<th>$x^2$</th>
<th>$\frac{1}{x}$</th>
<th>$\sin x$</th>
<th>Total$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(f(2 + h))^b$</td>
<td>80</td>
<td>20</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>$f(g(2 + h))^c$</td>
<td>0</td>
<td>50</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>$g(2) + h^d$</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Blank</td>
<td>6</td>
<td>31</td>
<td>63</td>
<td>100</td>
</tr>
<tr>
<td>Other$^e$</td>
<td>33</td>
<td>33</td>
<td>33</td>
<td>100</td>
</tr>
</tbody>
</table>

$^a$ These totals are accurate for actual percentages and not for reported nearest percent estimates.

$^b$ $f(2 + h)^2$, $\frac{1}{f(2 + h)}$, and $\sin (f(2 + h))$, respectively.

$^c$ $f((2 + h)^2)$, $f(\frac{1}{2 + h})$, and $f(\sin (2 + h))$, respectively.

$^d$ $2^2 + h$, $\frac{1}{2} + h$, $\sin 2 + h$, respectively.

$^e$ Although the distribution of errors are the same across function types for this error category, it is not the same group of students who made this type of error in each function type.

category $f(g(2 + h))$, were made with the rational and trigonometric functions. We can also see that all the errors in the category of $f(2) + h$ were of the form $\sin 2 + h$. In addition, over 50% of the answers were left blank for the trigonometric function. This supports previous statements that function type has an affect on ability to answer the question, and also on the type of error made.

Results from the statistical analysis, Table XIII, and Table XIV converged to show that function type affected the ability to evaluate $f(2 + h)$, while Table XV and Table XVI showed
that function type affected the type of error committed. In particular, the greatest distinctions
in error types, and ability to evaluate $f(2 + h)$ are between $x^2$ and $\sin x$.

4.2.3 Question 3: Evaluate $DQ_h$ (Quiz 2, Appendix A.2)

The errors made from answering Question 3 are the initial motivation of this study.¹ Questions 1 and 2 were asked in order to compare student solutions to solutions from Question 3. These comparisons will be made in the last subsection. In this subsection, I will discuss findings from the analysis of the errors found in student work on this question. These findings are presented in the same fashion as the previous two sections. That is, sample solutions are presented followed by analysis from multiple angles, triangulating to show that function type does not affect question score, but seems to affect which errors were committed. In addition, results from this section show that every student who got this question wrong (for any of the three function types) either did not evaluate $f(x + h)$, or evaluated it incorrectly. Some of these students also did not evaluate the $f(x)$ in the $DQ_h$ correctly.

Figure 9 is an example of correct student work that received a score of 3. For each function type, the student correctly evaluated $f(x+h)$ and also provided the correct $f(x)$ in the numerator of $DQ_h$. Figure 10 on the other hand shows two student samples, neither of which shows the correct evaluation of $f(x+h)$ or $f(x)$. Both student samples received scores of 0 on this question.

¹Despite that fact that Question 3 is the initial motivation of the study, it comes third because the organization of this chapter follows the content progression of the text, $DQ_h$ coming after $DQ_a$ and $f(2 + h)$.
3. Set up the difference quotient,
\[
\frac{f(x+h) - f(x)}{h},
\]
for each of the following functions:

(a) \( f(x) = x^2 \)
\[
\frac{(x+h)^2 - x^2}{h}
\]
(b) \( f(x) = \frac{1}{x} \)
\[
\frac{\left(\frac{1}{x+h}\right) - \frac{1}{x}}{h}
\]
(c) \( f(x) = \sin x \)
\[
\frac{\sin(x+h) - \sin x}{h}
\]

IV. Set up the difference quotient:
\[
\frac{f(x+h) - f(x)}{h}
\]
for each of the following functions:

1. \( f(x) = x^2 \)
\[
\frac{f(x+h) - f(x)}{h}
\]
2. \( f(x) = \frac{1}{x} \)
\[
\frac{\frac{1}{x+h} - \frac{1}{x}}{h}
\]
3. \( f(x) = \sin x \)
\[
\frac{\sin(x+h) - \sin x}{h}
\]
Let us take a look at the distribution of the quiz scores for each lecture. Figure 11 shows that over 70% of the students answered all three parts of this question correctly. The rest of the student scores fall into scores of 2 or lower; among these percentages, the highest in both lectures, 0 scores (13% and 16% respectively), suggests that when students had difficulty with this question, they had difficulty with all function types more frequently than with just one of the function types. This suggests that when students had difficulty with one function type, they also had difficulty with the other types. In the next paragraph, we take a closer look at the 2, 1, and 0 scores.
Table XVII details the percentage of students who got 0 on any of the three parts, resulting in a total question score of 2, 1, or 0. For example, 17% of all the monomial scores were 0 in Lecture A, and 16% in Lecture B. The majority of students did not get this problem wrong, but the percentages of students who got this problem wrong for any given function type is not low enough to ignore. The highest percentage of wrong answers appears in the trigonometric function for both lectures (19% and 24% respectively), but it is not clear if this percentage is high enough to be considered statistically significant compared to question scores on the monomial and rational functions.

A one way repeated measures ANOVA is employed to compare the effect of function types, monomial, rational, and trigonometric, on question score. Degrees of freedom were corrected using Green-Geisser estimates of sphericity ($\epsilon = .85$) because Mauchly’s test indicated that the assumption of sphericity had been violated ($\chi^2(2) = 19.61, p < .05$). The results show that there were no significant differences ($\alpha = .05$) in the scores of the three function types.
TABLE XVIII: PERCENTAGES OF ERRORS MADE IN EACH LECTURE FOR QUESTION 3

<table>
<thead>
<tr>
<th>Lecture</th>
<th>$f(x)$</th>
<th>$x^2$</th>
<th>$\frac{1}{x}$</th>
<th>$\sin x$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>34</td>
<td>32</td>
<td>34</td>
<td>100</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>28</td>
<td>34</td>
<td>38</td>
<td>100</td>
</tr>
</tbody>
</table>

$F(1.69, 167.59) = 1.98, p = .15$. In other words, function type does not have an effect on the ability to set up $DQ_h$.

The focus now shifts away from the question scores and onto the errors made on this question to investigate the effect of function type on these errors. Table XVIII shows the percentage of errors made for each function type out of total errors made in each lecture. For example, 34% of the total errors made in Lecture A were made with the monomial function. Lecture A is fairly split in percentage of errors across all three function types, supporting the findings from the ANOVA analysis. Lecture B diverges from Lecture A: 28% of the total errors were made with the monomial and 38% made with the trigonometric function.

All student work on Question 3 was parsed into categories and like the previous analyses, data from lectures are combined in Table XIX. Similar to the previous analyses of questions, the given $f(x)$ in the problem are renamed $g(x)$ for facility of discussion. There are several error categories for this question, beginning with students who left $f(x + h)$ as is on the quiz. More commonly, as seen on the left side of Figure 10, students gave the answer $f(x + h)^2$, i.e. used
the $f(x + h)$ given in the DQ$_h$ as an input to $g(x) = x^2$, so categorized as $g(f(x + h))$. Also seen on the left side of Figure 10, error III, students composing the opposite of the previously mentioned error: evaluating $g(x+h)$ and then using it as an input for $f(x)$, that is, $f(\frac{1}{x+h})$, and $f(\sin(x+h))$. The next category (IV), $g(x) + h$, is not demonstrated in the figure. Whatever the given $g(x)$, monomial, rational, or trigonometric function, students wrote $x^2 + h$, $\frac{1}{x} + h$, and $\sin x + h$ on their quizzes.¹ This error is used as an input to $f(x)$ in the category V, $f(g(x)+h)$, and can be found on the right side of Figure 10. Note that errors I–V are all problems with evaluating $f(x+h)$. Error categories from Question 1 can also be found in this question, namely, $g(f(x))$ and $f(g(x))$. These errors can be seen on both sides of Figure 10 where $f(x)$ is situated in the DQ$_h$: $f(x)^2$ on the left, and $f(x^2)$ on the right and $f(g(x))$ on both sides. The last error category in Table XIX consists of incomprehensible answers.

To give us insight into which error category is the most prominent for each function type, we look at the distribution of errors within each function type, for all error categories. In a later table, Table XX, we will look at which function types are most popular for each error type. Table XIX tells us that the most frequent error type for the $g(x) = x^2$ is $g(f(x+h)) = f(x+h)^2$ at 32%. However, for the rational function, $f(g(x)+h)) = f(\frac{1}{x+h})$ is the most popular error at 22%, and $g(x) + h = \sin x + h$ is the most popular error category for the trigonometric function at 28%. This tells us that function type affects the type of errors made in evaluating the DQ$_h$.

¹It is possible that students left out the parentheses, but the fact that this error was prominent across all function types suggests that the error is $g(x) + h$. 
TABLE XIX: PERCENTAGE OF ERROR DISTRIBUTION WITHIN EACH FUNCTION TYPE FOR QUESTION 3

<table>
<thead>
<tr>
<th>Error Category</th>
<th>$x^2$</th>
<th>$\frac{1}{x}$</th>
<th>$\sin x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. No $f(x + h)$ evaluation</td>
<td>14</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>II. $g(f(x + h))^a$</td>
<td>32</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>III. $f(g(x + h))^b$</td>
<td>0</td>
<td>22</td>
<td>16</td>
</tr>
<tr>
<td>IV. $g(x) + h^c$</td>
<td>5</td>
<td>13</td>
<td>28</td>
</tr>
<tr>
<td>V. $f(g(x) + h)^d$</td>
<td>9</td>
<td>17</td>
<td>12</td>
</tr>
<tr>
<td>VI. $g(f(x))^e$</td>
<td>9</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>VII. $f(g(x))^f$</td>
<td>14</td>
<td>17</td>
<td>12</td>
</tr>
<tr>
<td>VIII. Blank</td>
<td>9</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>VIII. Other</td>
<td>9</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td><strong>Total$^g</strong></td>
<td><strong>100</strong></td>
<td><strong>100</strong></td>
<td><strong>100</strong></td>
</tr>
</tbody>
</table>

$a$ $f(x + h)^2$, $\frac{1}{f(x + h)}$, and $\sin (f(x + h))$, respectively.

$b$ $f((x + h)^2)$, $f(\frac{1}{x + h})$, and $f(\sin (x + h))$, respectively.

$c$ $x^2 + h$, $\frac{1}{x} + h$, and $\sin x + h$, respectively.

$d$ $f(x^2 + h)$, $f(\frac{1}{x} + h)$, and $f(\sin x + h)$, respectively.

$e$ $f(x)^2$, $\frac{1}{f(x)}$, and $\sin (f(x))$, respectively.

$f$ $f(x^2)$, $f(\frac{1}{x})$, and $f(\sin x)$, respectively.

$^g$ These totals are accurate for actual percentages and not for reported nearest percent estimates.
Table XX looks at which function types are more likely to be present in each error type. For example, in the first error type, where students did not evaluate the $f(x + h)$, there is an equal percentage of students who committed the error (33%) for each function type. So for this error type, it did not seem to matter whether the function was $x^2$, $\frac{1}{x}$, or $\sin x$. However for error type II, students seem to only make this error with $x^2$. This was also the case for the $g(f(x)) = f(x)^2$ error. Interestingly, for the error type III, $f(g(x + h))$, students only made this error with the rational and trigonometric functions. In fact, with the exception of the types I, II, and VI, the monomial function either had the lowest percentage of errors, or tied as the lowest. This suggests that function type effects the type of error students make on this question.

Results from Figure 11 and the statistical analysis show that there are no significant differences in scores with respect to function type. However, Table XIX and Table XX show that function types affect which type of error was committed.

4.2.4 Question 4: Routine Function Composition (Quiz 3, Appendix A.3)

This routine composition question was included to compare scores to the scores of DQ questions, so it was designed to mimic the three function types of the previous three questions. That is, the $f(x)$ in each part of the problem is one of the three function types, and the $g(x)$ is in the form $x + h$.\(^1\) I explored how students generally performed on the question, including errors they made and the distribution of the scores. The errors and scores were examined to see if composition problems structured like $f(x + h)$ were problematic for students. Analysis of

\(^1\)A typo was made with the rational function on this question with one of the Lectures. Therefore, it is left out of this analysis and only results from the monomial and trigonometric functions are discussed.
TABLE XX: PERCENTAGE OF ERROR DISTRIBUTION ACROSS FUNCTION TYPES
FOR QUESTION 3

<table>
<thead>
<tr>
<th>Error Category</th>
<th>$g(x)$</th>
<th>$x^2$</th>
<th>$\frac{1}{x}$</th>
<th>$\sin x$</th>
<th>Totala</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. No $f(x + h)$ evaluation</td>
<td></td>
<td>33</td>
<td>33</td>
<td>33</td>
<td>100</td>
</tr>
<tr>
<td>II. $g(f(x + h))$b</td>
<td></td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>III. $f(g(x + h))c$</td>
<td></td>
<td>0</td>
<td>56</td>
<td>44</td>
<td>100</td>
</tr>
<tr>
<td>IV. $g(x) + h$d</td>
<td></td>
<td>9</td>
<td>27</td>
<td>64</td>
<td>100</td>
</tr>
<tr>
<td>V. $f(g(x) + h)e$</td>
<td></td>
<td>22</td>
<td>44</td>
<td>33</td>
<td>100</td>
</tr>
<tr>
<td>VI. $g(f(x))f$</td>
<td></td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>VII. $f(g(x))g$</td>
<td></td>
<td>30</td>
<td>40</td>
<td>30</td>
<td>100</td>
</tr>
<tr>
<td>VIII. Blank</td>
<td></td>
<td>29</td>
<td>29</td>
<td>43</td>
<td>100</td>
</tr>
<tr>
<td>VIII. Other</td>
<td></td>
<td>31</td>
<td>31</td>
<td>38</td>
<td>100</td>
</tr>
</tbody>
</table>

a These totals are accurate for actual percentages and not for reported nearest percent estimates.

b $f(x + h)^2$, $\frac{1}{f(x + h)}$, and $\sin (f(x + h))$, respectively.

c $f((x + h)^2)$, $f\left(\frac{1}{x + h}\right)$, and $f(\sin (x + h))$, respectively.

d $x^2 + h$, $\frac{1}{x} + h$, and $\sin x + h$, respectively.

e $f(x^2 + h)$, $f\left(\frac{1}{x} + h\right)$, and $f(\sin x + h)$, respectively.

f $f(x)^2$, $\frac{1}{f(x)}$, and $\sin (f(x))$, respectively.

g $f(x^2)$, $f\left(\frac{1}{x}\right)$, and $f(\sin x)$, respectively.
student work on this question show that there are fewer error categories and higher scores than in the DQ questions. These findings motivate the possible use of composition as a teaching aid in DQ evaluation. Further discussion of this can be found in the next chapter in the qualitative portion of the study.

Figure 12 shows an example of student work which received a total score of 2; one point for part a and one for part c. A group of students used the chain rule to answer this question. Although that is not the correct answer, it was accepted because it indicated the ability to compose two functions. Figure 13 shows an example of a student who composed $f'$ and $g$ as required in the chain rule. However, if a student attempted use the chain rule and did so incorrectly, it was marked as a 0. Figure 14 is an example of a such a quiz. Besides taking an incorrect derivative, there were only a few error categories and there were no patterns that existed among them.

A paired-samples t-test was conducted to compare the question scores in monomial and trigonometric function types. There was not a significant difference in the scores for the monomial ($\mu = .89, SD = .31$) and trigonometric ($\mu = .86, SD = .35$) types; $t(92) = .90, p = .37$. This suggests that function type did not affect the ability to answer the question.

Figure 15 shows Lectures A and B diverging in the distributions of total quiz scores. In Lecture A, 16% got one correct and one wrong, while one fourth of that percentage, 4% got both parts of the question wrong. In Lecture B, these percentages were much closer to each

---

1As a result of the typo, the maximum score possible on this question was 2.
1. Compose $f$ and $g$, i.e. give $f \circ g(x) = f(g(x))$, for: (you do not need to simplify your answers)

(a) $f(x) = x^3$ and $g(x) = x + 1$

$$f(g(x)) = (x+1)^3$$

(c) $f(x) = \tan x$ and $g(x) = x - 1$

$$f(g(x)) = \tan(x-1)$$
Figure 13: Sample of Chain Rule Student Work for Question 4

1. Compose $f$ and $g$, i.e. give $f \circ g(x) = f(g(x))$, for: (you do not need to simplify your answers)

(a) $f(x) = x^3$ and $g(x) = x + 1$

$f'(x) = 3x^2$ 
$g'(x) = 1$

$f \circ g(x) = [3(x+1)^2] \cdot 1$

(b) $f(x) = \tan x$ and $g(x) = x - 1$

$f'(x) = \sec^2 x$ 
$g'(x) = 1$

$f \circ g(x) = [\sec^2 (x-1)] \cdot 1$
other; 10% got one out of two of the parts correct, and 7% got both parts wrong. Explanations of why these differences occurred between these lectures is beyond the scope of this study. The most important information this figure conveys is that overall, students did not have difficulty with this problem. Despite the divergence in non-perfect scores, the percentages of perfect scores in each lecture, 80% and 83%, are fairly close to each other.

Table XXI shows the percentage of students (out of the total number of students) who scored 0 for any of the two function types, thereby receiving a total quiz score of either 1 or 0. For example, 10% of the total scores for the monomial function type were 0 in Lecture A and 14% of the total scores for the trigonometric function were 0. This suggests that students seem to get
Figure 15: Scores for Question 4 by Lecture

<table>
<thead>
<tr>
<th>Lecture</th>
<th>Score 2</th>
<th>Score 1</th>
<th>Score 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>41</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>35</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>
TABLE XXI: PERCENTAGES OF STUDENTS IN EACH LECTURE WHO SCORED 0 FOR QUESTION 4

<table>
<thead>
<tr>
<th>Lecture</th>
<th>$x^2$</th>
<th>$\sin x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>B</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

this question incorrect for $\sin x$ more frequently than for $x^2$, while in Lecture B the distribution of 0 scores across function types are even at 12%. This is a similar finding from the previous paragraph; that is, there seems to be a difference in scores by lecture. We see this in the next table as well.

In Table XXII we move away from the scores and focus on the number of errors made with each part of the question. For example, 42% of the total errors made in Lecture A were made with $x^2$, and 58% were made with $\sin x$. Lecture A seems to have made more errors with $\sin x$ than with $x^2$, whereas the percentage of errors made in Lecture B were split evenly between $x^2$ and $\sin x$. Lecture B is a closer reflection of the statistical analysis where it was shown that there were no statistical differences to suggest an effect of function type on composing two functions.

There were very few errors made in this question, and even fewer error categories. The most significant error was attempting to apply the chain rule to answer the question, but failing to correctly do so. The other errors—which included multiplying $f(x)$ with $g(x)$ and blank answers—did not reveal any significant patterns. Since the numbers are small, the raw data for
TABLE XXII: PERCENTAGES OF ERRORS MADE IN EACH LECTURE FOR QUESTION 4

<table>
<thead>
<tr>
<th>Lecture</th>
<th>$x^2$</th>
<th>$\sin x$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>43</td>
<td>57</td>
<td>100</td>
</tr>
<tr>
<td>B</td>
<td>50</td>
<td>50</td>
<td>100</td>
</tr>
</tbody>
</table>

TABLE XXIII: ACTUAL ERROR DISTRIBUTION WITHIN EACH FUNCTION TYPE FOR QUESTION 4

<table>
<thead>
<tr>
<th>Error Category</th>
<th>$x^2$</th>
<th>$\sin x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Took some derivative</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Other</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>11</strong></td>
<td><strong>13</strong></td>
</tr>
</tbody>
</table>

The error types are presented here in Table XXIII. Looking across function types in the same error category, there isn’t a significant different in the raw numbers. The same is true across error types in the same function type. There does not seem to be an effect in function type on error type.

The statistical analysis and Table XXIII tell us that there are no significant differences in scores with respect to function type. Table XXIII also tells that function type does not seem to affect the type of error committed. When we look at the lectures separately from each other, the
distribution of the errors and the 0 scores do not seem to reflect each other. However, Lecture B does reflect the results that are pooled together.

### 4.2.5 All Questions

This section reports the trends from the findings across all four of the quiz questions. These trends include common error types that spanned several questions. Relationships between all four quiz scores are also presented in this section.

In the previous four subsections, error categories for each question were identified. This information told us where students struggle with the important concept of the difference quotients. In particular, for the $DQ_a$ question, the error categories showed that students had trouble with evaluating $f(x)$, $f(a)$, and $x$ in the denominator. In the second question, $f(2+h)$ evaluation was most problematic for $\sin x$. Evaluation of $f(x+h)$ and $f(x)$ gave students the most difficulty in the $DQ_h$. For the last problem, students who made errors either took an incorrect derivative or made an ungeneralizable error.

Table XI (p. 47), Table XV (p. 57), and Table XIX (p. 65) showed error categories for Question 1, Question 2, and Question 3, respectively. There are trends in error types across the $DQ_a$, $f(2+h)$ evaluation, and $DQ_h$ questions. Table XXIV shows the errors that were present in more than one question and what these errors look like for each question. For function type $g(x) = x^2$, $g(f(x))$ is the common error across all three questions, namely $f(x)^2$, $f(2+h)^2$, and $f(x+h)^2$, respectively. For $g(x) = \frac{1}{x}$, it is the $f(g(x))$ error type, i.e. $f\left(\frac{1}{x}\right)$, $f\left(\frac{1}{2+h}\right)$, and $f\left(\frac{1}{x+h}\right)$, respectively. For $g(x) = \sin x$, the common error category was also $f(g(x))$: $f(\sin x)$, $f(\sin (2+h))$, $f(\sin (x+h))$. In addition, the $f(x)$ in both the $DQ$’s surfaced the same error.
types: \( f(g(x)) \) and \( g(f(x)) \). For \( f(2 + h) \) evaluation and \( f(x + h) \) evaluation, \( \sin x \) also caused the same error, \( f(x) + h \), i.e. \( \sin 2 + h \) and \( \sin x + h \). Common error types across different categories tells us that these errors span different contexts.

An analysis of student membership in error categories across different questions tells us that there was little overlap of students in a particular error type for one question and those in the same error category for a different question.\(^1\) In fact, 73 students out of the total 123 participants made at least one error on at least one question for any given quiz. Of these 73, only five students made the same error for at least two of the questions. In other words, these error types affect many students. As will be shown in Chapter 5, this is reflected in the interviews; students who had not previously made a mistake on their quiz would make the error during the interview. This fact, along with the fact that common errors span different contexts, signals the importance of studying the reasons students are making these errors.

In this section, the relationship between student scores on the four questions are presented. Since the fourth question contained a typo for the rational function, only the scores for the monomial and the trig function were analyzed. Table XXV shows the correlations between the student scores on the monomial part of each question. The scores for DQ_a and \( f(2+h) \), and DQ_a and DQ_h are significantly correlated at the .01 and .05 level, \( r(107) = .242 \) and \( r(96) = .373 \), respectively. There is also a significant correlation at the .05 level between the scores of the composition question and the DQ_h, \( r(83) = .245 \).

\(^1\)The scores of the three quizzes were linked to each other through the student that took the quizzes. In other words, given Student A, it is possible to look up his/her scores on each of the questions.
### TABLE XXIV: SHARED ERROR CATEGORIES ACROSS THE FIRST THREE QUESTIONS

<table>
<thead>
<tr>
<th>Function Type</th>
<th>Error Category</th>
<th>1. $\text{DQ}_a$</th>
<th>2. $f(2 + h)$</th>
<th>3. $\text{DQ}_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>$g(f(x))$: $f(x)^2$</td>
<td>$f(x + h)$</td>
<td>$f(x)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$f(g(x))$: $f(x^2)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$f(g(x))$: $\frac{1}{x}$</td>
<td>$\frac{1}{2 + h}$</td>
<td>$\frac{1}{x + h}$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$f(g(x))$: $f(\sin x)$</td>
<td>$f(\sin (2 + h))$</td>
<td>$f(\sin (x + h))$</td>
<td>$f(\sin x)$</td>
</tr>
<tr>
<td></td>
<td>$g(x) + h$: $\sin 2 + h$</td>
<td>$\sin x + h$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### TABLE XXV: CORRELATIONS OF QUESTION SCORES FOR MONOMIAL FUNCTION TYPE

<table>
<thead>
<tr>
<th></th>
<th>$\text{DQ}_a$</th>
<th>$f(2 + h)$</th>
<th>$\text{DQ}_h$</th>
<th>$f(g(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{DQ}_a$</td>
<td>1</td>
<td>.242*</td>
<td>.373**</td>
<td>.159</td>
</tr>
<tr>
<td>$f(2 + h)$</td>
<td>.242*</td>
<td>1</td>
<td>.144</td>
<td>.021</td>
</tr>
<tr>
<td>$\text{DQ}_h$</td>
<td>.373**</td>
<td>.144</td>
<td>1</td>
<td>.245*</td>
</tr>
<tr>
<td>$f(g(x))$</td>
<td>.159</td>
<td>.021</td>
<td>.245*</td>
<td>1</td>
</tr>
</tbody>
</table>

* Correlation significant at the .05 level (2-tailed).
** Correlation significant at the .01 level (2-tailed).
TABLE XXVI: CORRELATIONS OF QUESTION SCORES FOR TRIGONOMETRIC FUNCTION TYPE

|       | \( DQ_a \) | \( f(2 + h) \) | \( DQ_h \) | \( DQ_a \) | \( f(2 + h) \) |
|-------|-------------|----------------|-------------|-------------|
| \( DQ_a \) | 1           | .379**         | .407**      | .165        |               |
| \( f(2 + h) \) |             | 1              | .330**      | .096        | .194         |
| \( DQ_h \) |             |                | 1           |             |               |

** Correlation significant at the .01 level (2-tailed).

For the trigonometric function type, there are significant correlations at the .01 level between the scores for \( DQ_a \) and \( f(2 + h) \), the \( DQ_a \) and \( DQ_h \), and the \( DQ_h \) and \( f(2 + h) \), \( r(107) = .379 \), \( r(96) = .407 \), and \( r(96) = .330 \), respectively. What is notable about this analysis is that there is no significant relationship between the composition question and any of the other three questions. In other words, there is no pattern in how students perform on composition questions compared with how they perform on the \( DQ \)'s and \( f(2 + h) \) evaluation.

4.3 Conclusion

The main purpose of this phase was to reveal error categories in \( DQ_a \) and \( DQ_h \). In my investigations, I have also surfaced errors in evaluating \( f(2 + h) \). This additional analyses was included because \( f(2 + h) \) is in the form \( f(x + h) \) for \( x = 2 \). In addition, I explored errors made with routine function composition questions in a format similar to \( f(x + h) \), i.e., for given function type \( f(x) \), and \( g(x) = x + a \) for some \( a \). This question was included to discover the
students’ perceived relationships between the evaluations of \( f(x + h) \) and function composition through quiz scores.

The following is a list of summary points from the quantitative phase of the study:

- Of the four questions, DQ\(_a\) evaluation had the lowest mean score, and the highest percentage of wrong answers.
- DQ\(_a\) had 6 major error categories: 2 for \( f(x) \) evaluation, 2 for \( f(a) \) evaluation, 1 for denominator evaluation, and 1 for people who used DQ\(_h\).
- Of the four questions, DQ\(_h\) had the most major error categories\(^1\) (7), while the composition had the least (1).
- There were 3 major error categories identified for \( f(2 + h) \) and 5 error categories for \( f(x + h) \) in the DQ\(_h\).
- Every quiz that received a 0 for any of the three parts of the DQ\(_h\) question involved failure to compute or correctly compute \( f(x + h) \).
- When students were asked to evaluate DQ\(_a\), some used DQ\(_h\) instead. This was not true the other way around.
- Different function types (monomial, rational, and trigonometric) *do not* affect students’ abilities to set up the DQ\(_a\), DQ\(_h\), or to compose standard function composition questions in a format similar to \( f(x + h) \).

\(^1\)Not including Blank or Other.
Different function types (monomial, rational, and trigonometric) do affect students’ abilities to set up \( f(2 + h) \), particularly for monomials and trigonometric functions.

For evaluation of \( DQ_a, f(2 + h), \) and \( DQ_b \), function type affects the type of error committed. Therefore for these three questions, errors cannot be categorized in a way that is independent of the type of function.

For the composition question, errors can be categorized independent of function type.

Same error categories were present across different questions. There was little to no overlap of students who performed the same errors across different questions. These errors are widespread in different contexts relating to function, and affect different students from question to question.

The errors that were identified in this phase of the study are a basis for the next phase. Four were selected for an in-depth analysis to investigate what these errors reveal about student thinking. This is detailed in the next chapter.
CHAPTER 5

PHASE 2: QUALITATIVE RESEARCH

5.1 Research Design

The purpose of this phase is to answer Research Question 2 (Section 1.3). The focus will be on the errors that were identified during the first phase of the study. The chapter begins with the research method used, followed by the selection, recruitment, and descriptions of the participants, and then details the data collection and analysis procedures. In Section 5.2 student thinking as revealed by conversations surrounding error categories will be discussed.

5.1.1 Research Method

The method of case study is used to answer Research Question 2. The focus is on the errors and what they reveal about students' thinking on functions and function composition when evaluating $DQ_h$ and $DQ_a$. Thus, the phenomena of interest are the errors. To learn about the complexities of these errors, students were interviewed because they are instrumental to relaying their conceptions of function, composition, and DQ evaluations.

The focus of this study is on the particularities and the complexities (Stake, 1995) of student errors, so the cases themselves are a subset of errors found in Phase 1. In Phase 2 of the study, four were chosen to be the units of analysis (Yin, 2009), or objects of study (Stake, 1995): No $f(a)$ Evaluation, $f(g(x))$ Error, $g(f(x))$ error, and Denominator Error. In each of these four cases, students were asked to comment on the quizzes with student errors, whether they
were their own, or belonged to someone else. Triangulating data from students who performed the errors with those who did not creates a robust image of student conceptions of function, composition, and DQ evaluations.

Guides for Interview 1 (Appendix B.1) and Interview 2 (Appendix B.2) were developed with another investigator and proofread by a team of mathematicians and mathematics educators. Though the guides were developed for standardization, students responded freely. Due to this freedom, interesting questions emerged in the initial interviews and consequently were included in future interviews (Rossman and Rallis, 2003).

Questions in the interviews were guided by theoretical analysis of previous work and also by subject matter (Zazkis and Hazzan, 1999). The purpose of each question, also developed and proofread by the aforementioned team, can be found in Appendix C. In the interviews, students were asked to perform and construct mathematical tasks. When appropriate, they were asked “give an example” tasks. During a portion of Interview 2, students were asked to reflect on other students’ work (Appendix D). The nature of this descriptive study to reveal student thinking required me to ask many unexpected “why” questions.

5.1.2 Selection, Recruitment, and Description of Participants

All participants from Lecture A and Lecture B of Phase 1 were invited to participate in Phase 2 of the study. I first introduced myself and my study to both lectures early in the semester. As part of the consent process in Phase 1, I also presented myself and the study in each of the lectures’ three respective teaching-assistant-led discussion sections (six total). After
Phase 1, I attended each lecture again to explain the qualitative phase of the study and to invite students to participate. Ten students volunteered to participate in the study.

The first five students in Table XXVII were from Lecture A and the last five come from Lecture B. The course grades reported by the instructors at the end of the semester were two A’s, one B, three C’s, two D’s, one F, and one student opted not to release her grade. There were two self-identified male participants and eight self-identified females. Both of the males self-identified as White. Of the females, there were four self-identified Whites, two self-identified Blacks, one self-identified Hispanic and one no response. Two participants identified themselves as Mexican, two Polish, two Irish, one Middle-Eastern, three African American, one German, and one Scottish.¹ There were three freshman, three sophomores, three juniors and one non-degree student getting his endorsement to teach math. Only one participant was a math major, one was an architecture student, and the rest were scattered among the sciences. Four of the students were experiencing Calculus for the second time. Interestingly, these participants all reported at least one of these categories: honors math in high school, good in math, or enjoy math. In general, the participants found the study of math favorable.

¹These numbers do not add up to ten since students could be members of multiple races.
<table>
<thead>
<tr>
<th>Name</th>
<th>Course</th>
<th>Grade</th>
<th>Gender</th>
<th>Ethnicity/Race</th>
<th>Year of Study</th>
<th>Major</th>
<th>Calculus Experience</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian</td>
<td>(C)</td>
<td>F</td>
<td>Hispanic (Mexican/Polish)</td>
<td>Junior</td>
<td>Kinesiology</td>
<td>1st time</td>
<td></td>
</tr>
<tr>
<td>Pierre</td>
<td>(B)</td>
<td>M</td>
<td>White (Irish)</td>
<td>Sophomore</td>
<td>Undeclared (Mechanical Eng.)</td>
<td>1st time</td>
<td></td>
</tr>
<tr>
<td>Ray</td>
<td>(A)</td>
<td>M</td>
<td>White (Polish)</td>
<td>Non-Degree</td>
<td>Teaching of History (Math)</td>
<td>1st time</td>
<td></td>
</tr>
<tr>
<td>Lauri</td>
<td>(C)</td>
<td>F</td>
<td>White (Middle-Eastern)</td>
<td>Sophomore</td>
<td>Biology</td>
<td>1st time</td>
<td></td>
</tr>
<tr>
<td>Drew</td>
<td>(D)</td>
<td>F</td>
<td>White (Mexican)</td>
<td>Freshman</td>
<td>Architecture</td>
<td>2nd time</td>
<td></td>
</tr>
<tr>
<td>Turner</td>
<td>(C)</td>
<td>F</td>
<td>(African American)</td>
<td>Sophomore</td>
<td>Undeclared (Math, Chem, Bio)</td>
<td>2nd time</td>
<td></td>
</tr>
<tr>
<td>Jo</td>
<td>(A)</td>
<td>F</td>
<td>Black (African American)</td>
<td>Freshman</td>
<td>Mathematics</td>
<td>1st time</td>
<td></td>
</tr>
<tr>
<td>Nat</td>
<td>(–)</td>
<td>F</td>
<td>Black (African American)</td>
<td>Junior</td>
<td>Biology</td>
<td>1st time</td>
<td></td>
</tr>
<tr>
<td>Sterne</td>
<td>(F)</td>
<td>F</td>
<td>White (Irish)</td>
<td>Junior</td>
<td>Undeclared (Criminal Justice)</td>
<td>2nd time</td>
<td></td>
</tr>
<tr>
<td>Hudson</td>
<td>(D)</td>
<td>F</td>
<td>White (German/Scottish)</td>
<td>Freshman</td>
<td>Biology (Business)</td>
<td>2nd time</td>
<td></td>
</tr>
</tbody>
</table>

*a All names have been changed.
The participants’ quiz scores are presented in Figure 16. All participants answered all four questions. The total scores ranged from 4 (Hudson) to 12 (Lauri, Drew, Jo, and Nat). The mean is 9.4, the median is 10 and the mode is 12.

5.1.3 Data Collection Procedures

During weeks 11 and 12 of Fall Semester 2010, each participant was interviewed twice for approximately 45-60 minutes. A total of 20 interviews were recorded. The first interview (Appendix B.1) was intended to gather information about the students’ backgrounds and to ease them into the interview process. In this interview they were also asked to define concepts...
of function and to give examples. They also commented on function notation and were asked to compare their comments to their conceptions of \( f(x + h) \). In the end, they were asked to perform labeling tasks on the graph of an arbitrary continuous function.

The second interview (Appendix B.2) was more in-depth and students’ thinking on concepts were probed with reflection questions and unexpected “why” questions. In this interview, students gave their understandings of function composition, what mathematical concepts use composition, and examples of composition. They were also asked to construct a function in a question that combined \( f(x + h) \) evaluation and function composition. A major portion of the interview was devoted to students' discussions on their quiz work and other student quiz work.

The Echo Pen and its accompanying notebook by LiveScribe were used to record all interviews. This data collection tool records spoken language while simultaneously documenting written language in the notebook. The recording of the interview shows a dynamic replay of the written work and its associated explanations. The audio can be extracted and saved as an audio file and the work in the LiveScribe notebook can be exported as pdfs. After all the participants were interviewed, the audio files were transcribed using the open source program Transcriptions.

5.1.4 Data Analysis Procedures

The interview transcripts, the audio, the pdfs of the notebook, the joint audio and visual files of the interviews, and the participants’ quizzes were all used during data analysis. Each transcript was initially read over fully before analysis. Then each transcript was sectioned off into topics or subtopics as denoted in the interview guides, and called episodes. For example, in
Interview I, the section, “Definition Questions and Functions,” was identified in each participant’s transcript. Then all the student excerpts pertaining to the episodes were pooled together for detailed analysis. For each topic/episode, every interview was read, listened to and watched, and then re-read and coded and re-coded several times. From this point, the analysis of data had two directions.

The one direction of the data analysis aims to minimize the gap between the intended theory and the enacted theory by contributing data analysis tools that unify general operational definitions, transition levels, strength indicators, definitions for each stage of conception that apply to specific topics of interest (i.e. related subtopics of functions), and raw interview data. To achieve this, I used the data from the interviews to appropriately match the raw data to the topic-specific definitions that came directly from the unified definitions provided by Chapter 2. The resulting data analysis tables can be found in Chapter 6 where the findings from Research Question 3 are presented.

The other focus of analysis is dedicated to investigating students’ conceptions of function, and DQα and DQβ evaluation as revealed by conversations about the errors that were previously identified in the quantitative phase of the study. The episodes where students discussed their errors, or their peers’ errors, were thoroughly analyzed for patterns of thinking associated with the errors. This analysis intends to inform the teaching and learning of students regarding early calculus material. The findings are presented in the Sections 5.2 and 5.3.
5.2 Findings from Intra-Case Analysis

In this case study, four of the error categories discovered from the quantitative phase of the study are chosen to be units of analysis:

**Case 1:** No $f(a)$ evaluation,

**Case 2:** $f(g(x))$ Error $[f(x^2), f(\frac{1}{x}), f(\sin x)]$,

**Case 3:** $g(f(x))$ Error $[f(x)^2, \frac{1}{f(x)}, \sin f(x)]$, and

**Case 4:** Denominator Error $[x^2 - a, \frac{1}{x} - a, \sin x - a]$.

The $f(a)$ error was chosen for in-depth analysis because nearly half of the participants either did not evaluate it, or they did so incorrectly. The next two errors were chosen for their prominence across the first three questions. Lastly, the denominator error was analyzed for its relationship to the second and third error. It is important to study these areas of error because they reveal patterns of student thinking, which in turn, inform educators to the reasons why students struggle with DQ evaluations. In the following subsections, I will present student discussions regarding the error categories and the general patterns among the group of students who performed the error.

Some of the students performed these errors on the quizzes, but others who had not, performed them during the interviews. It is important to note that in these discussions, these errors (that were found and categorized in the first phase of the study) presented themselves during the interviews before I showed the participants the samples of the errors. In other words, these
errors were not limited to those who performed them on the quizzes. This suggests that they are more problematic than what the data analysis from Phase 1 showed.

5.2.1 Case 1: No $f(a)$ Evaluation (Lian, Pierre, Ray, Hudson)

Nearly 50% of all students from Phase 1 did not evaluate $f(a)$ in $DQ_a$ on their quizzes, or did so incorrectly. Phase 2 participant Ray is among this group of students. In Ray’s second interview, he reevaluated the $DQ_a$ for $x^2$ after I showed him his quiz (Appendix E.5). Though he gave an answer different from his original, he still did not evaluate $f(a)$ because “it’s” not given. Below he talks about the numerator of the $DQ_a$, $f(x) - f(a)$:

Ray: ...So this would be $x^2$ minus the function of $a$. Which we do not know, or do we? [6 second pause]. No...#00:16:12.7#

G: ...so you don’t know what $f(a)$ is? #00:17:05.9#
Ray: No. #00:17:07.8#
G: Why not? #00:17:07.3#
Ray: It’s not given to us. #00:17:09.8#

It is unclear if the “it” he referred to is $f(a)$ or $a$. If the concern is whether $a$ is given or not, then it would suggest that he is looking for a numerical value for $a$ so that he could evaluate $f(x)$ at that number, signaling a relationship between $f(a)$ and $f(x)$ [action conception]. In contrast, if the concern is whether $f(a)$ is given, then it suggests that he believes that $f(a)$ and $f(x)$ are unrelated [pre-action conception].

\[^{1}\text{Misconceptions are classified as pre-action conceptions. This conception stage is discussed in Chapter 6.}\]
Like Ray, Pierre also could not give $f(a)$ during Interview 2 when asked to describe his work on the quiz. He describes his work on evaluating the DQ$_a$ for $f(x) = \frac{1}{x}$ below.

Pierre: ...I know my $f(x) = \frac{1}{x}$, so I plug that in there, $\frac{1}{x}$, and I wasn’t given an exact point [writes $\frac{1}{x} - f(a)$, but writes $f(a)$ first and then puts the parenthesis around $a$ last]. But if I was—that would make it easy. #00:09:54.8#
G: Wait, so you can’t give $f(a)$ right now? #00:09:57.4#
Pierre: [6 seconds pass] Maybe. Let me look at this. [4 seconds pass] No. #00:10:08.5#
G: Ok. #00:10:10.0#
Pierre: No. #00:10:10.4#

:;
Pierre: Yeah so, if I had $f(a)$, I could find the exact derivative ... #00:10:23.6#
:;
Pierre: ...It’s basically, just how you find a difference quotient, or a derivative, if you have a function, $f(x)$, and you have a point, $f(a)$, then you can just plug it in there and find what the derivative is. #00:11:28.9#

This dialog shows that “point” that he kept referring to is $f(a)$, not $a$. Pierre exhibits a pre-action conception because he views $f(a)$ as disconnected from $f(x)$; he wanted $f(a)$ to be given as opposed to $a$.

Similar to Pierre, Lian did not view $f(a)$ as related to $f(x)$ because she viewed $f(a)$ as the “what” that is not given, rather than $a$. On her quiz, Lian used the DQ$_h$ instead of the DQ$_a$ to answer Quiz Question 1, so I asked her to redo the problem with the DQ$_a$ for each function type beginning with $x^2$ and she responded:

Lian: [Writes $f(x) = x^2$]. [14 seconds pass]. I don’t know, I don’t know what $f(a)$ is. You just write it out? #00:15:27.2#
G: Sure, just answer it how it how you think it should be. [Student writes $\frac{x^2 - f(a)}{x^2 - a}$].

Ok, can now—can you explain each part to me please? #00:15:43.3#

Lian: I just left it minus $f(a)$, since it wasn’t given? #00:16:11.0#

G: What is not given? #00:16:12.6#

Lian: $f(a)$. #00:16:14.3#

G: So then... #00:16:16.4#

Lian: Unless there is a way to find it, but I don’t remember. #00:16:19.3#

She did not view $f(a)$ as related to $f(x)$, so how does she view the relationship between $f(a \text{ number})$ and $f(x)$? Below I asked her:

G: Ok. What if it said, $f(2)$? Then what would you do? #00:16:28.9#

Lian: Plug in 2 to the $f(x)$. #00:16:31.3#

G: But, so since— #00:16:34.4#

Lian: Oh, so you would substitute it; I see what you're saying [writes $\frac{x^2 - a^2}{x^2 - a}$]. #00:16:46.4#

The words $f(2)$ conjured up the word “plug”. Lian’s evaluation of $f(2)$ was completely imagined so she must have a process view of $f(a \text{ number})$. However, she did need an external cue for $f(a)$, so she has an action conception of $f(a)$. Note that she has different conceptions of $f(a \text{ number})$ and of $f(a)$.

During the $f(x + h)$ conversation in Interview 1 with Hudson, she could not evaluate $f(a)$. Here’s an excerpt of that dialog.

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1She made the denominator error for this question as well. Detailed discussion of this error can be found in the Section 5.2.4.

2In Interview 1, Hudson commented that $f(x)$ and $f(x + h)$ were unrelated. In an attempt to understand her (mis)conception of $f(x + h)$, I investigated her understanding of $f(a)$. For details, see Interview 2 beginning after #00:15:49.8# and ending #00:17:45.8# (Appendix ??).
G: ...for this \( f(x) = x^3 + 2x^2 \), say I were to ask you to find—write what \( f(a) \) would be. For that one, what would you say? #00:18:03.0#
Hudson: I would have no idea how to do that. #00:18:06.5#
G: So if I said... #00:18:07.3#
Hudson: \( a \) is nowhere near this problem. #00:18:10.5#
G: So if I said, ok, if you have this one. If you have this \([f(x) = x^3 + 2x^2]\). What would \( f(a) \) be? What would you say? #00:18:21.3#
Hudson: I would say, you don’t have enough information. #00:18:24.8#

Hudson was clearly focused on the absence of \( a \) rather than \( f(a) \). Unlike Pierre and Lian, she understood that \( f(a) \) is connected to \( f(x) \), but did not understand what is \( a \). To see what her conception of \( f(a \text{ number}) \) was compared to \( f(x) \), I gave Hudson the same external cue that I gave Lian:

G: Ok, what if I said, what is \( f(2) \)? What would you say? #00:18:31.3#
Hudson: Oh ok, \( a \) can just, it’s just like another variable. So like \( f(2) \) would be \([\text{writes } 8 + 8 = 16]\) that. So then \( f(a) \) would be like. \( a^3 + 2a^2 \). #00:18:56.3#

Hudson and Lian both required an external cue, \( f(2) \) to give \( f(a) \). Thus, they both only have action conceptions of \( f(a) \) because they “can carry out the transformation only by reacting to external cues that give precise details on what steps to take” (Asiala et al., 1996, p. 7). Lian used the word plug while Hudson did not, but Lian did not have to explicitly calculate \( f(2) \), while Hudson did before giving \( f(a) \). As such, Hudson is still restricted to an action conception of \( f(2) \).

On her quiz, Hudson had used the quotient rule rather than the DQ\( a \). When asked to redo the problem during Interview 2, Hudson was able to evaluate \( f(a) \) with none of problems that she exhibited in Interview 1.
Pierre and Ray were not able to evaluate the $f(a)$ on their quizzes or during the interviews, while external cues triggered Lian and Hudson’s evaluations of $f(a)$. An analysis of the time it took for the eight students who evaluated $f(a)$ to give $a^2$, $\frac{1}{a}$, or $\sin a$ showed that Lian, Sterne and Lauri paused for 6 seconds every time before this evaluation, while the others had no hesitations. Note that the others (Drew, Turner, Jo, and Nat) also happen to be the students that did not make any of the errors analyzed in this case study.\(^1\) In addition, for Pierre and Lauri, when they wrote $f(a)$ during the interviews, they wrote it differently than they wrote $f(x)$; they both wrote $fa$, then filled in the parentheses last. Writing $f(a)$ in this order masks $a$ as an input value of $f(x)$ and is what I would consider as a pre-action view because it does not promote even an action conception of a function. Furthermore, this is evidence that for Pierre $f(x)$ and $f(a)$ are two separate concepts. This is also supported in the mismatch in Pierre’s conception of function (at most object) and his conception of $f(a)$ (at most pre-action).

Note the final grades each of the four students who did not originally evaluate $f(a)$ received: A, B, C, D. This tells us that this kind of error afflicts students from many levels and is not particular to students who struggle with Calculus. In fact, this subsection has shown that even for students who evaluated $f(a)$ correctly on the quiz, doing so is not a trivial task. For students who struggled with $f(a)$, they exhibited pre-action conceptions of $f(a)$ because it is seen as disconnected from $f(x)$. Thus, conceptions of $f(a)$ and $f(x)$ may not match. However, giving external clues helped move them into an action conception.

\(^1\)Turner is among the group of students who gave $f(a)$ as $a$, but only for the rational function type (Appendix E.11). This was the only error made on any of her quizzes.
5.2.2 Case 2: \( f(g(x)) [f(x^2), f(1/x), f(\sin x)] \) Error (Lian, Pierre, Ray, Lauri)

As shown in Table XXIV from Phase 1, this error was prevalent across the DQ\(_a\), \( f(2 + h) \), and DQ\(_h\) evaluation questions, suggesting that this error is not isolated to a particular type of calculus problem. Nor is it particular to certain students; as Phase 1 analysis showed, the errors were not necessarily repeated by the same students across quizzes. As such, there was no pattern connecting students performing the error from one question to another. In addition, some students did not make this mistake on their quizzes, but they did make it during their interviews. The prominence of this error across different questions and among different students, along with its ability to affect students whose quiz scores would suggest otherwise, underscore the need for investigating student thinking as revealed by this error.

Of the four interviewees who are members of this error category, only Ray made this mistake on the quiz; the others made it during the interview. Ray made this mistake for every function type on his quiz for DQ\(_a\), i.e., for the \( f(x) \) in the numerator of the DQ\(_a\), he wrote \( f(x^2), f(\frac{1}{x}), f(\sin x) \), respectively. However, during the interview when he reevaluated the DQ\(_a\), he did not make the mistake. When asked why he initially made it on the quiz, regarding \( x^2 \) he said, “on the quiz, I just plugged in the \( x^2 \) [circles \( x^2 \)], where it said \( x \) in the function of \( x \) right [writes \( f(x) \)]? . . . I wasn’t, I guess I was just thinking, not straight” (Interview 2, #00:16:55.6#) I asked him why again in a later conversation and he said, “I honestly don’t know. That [substitute] TA just told us to plug them in, that’s what I did,” (Interview 2, #00:21:59.5#). Ray indicates that his error is a result of his interpretation of the instructor’s directions. This becomes a theme
among the other errors as well. Discussion is left for the final section that generalizes across all four errors.

For Lian, this error was not on the quiz, but came up twice: once during our discussion about $f(x + h)$ in Interview 1, and again during Interview 2 when we were discussing $DQ_h$. Our discussion on this topic is long and confusing without seeing the work that she writes, so it is not cited here in its entirety. Instead, I summarize our conversation with short quotes from Lian.

In our conversation, Lian used the equal sign as an instruction of what to do with the expressions on either side of the symbol rather than using it to relate the two expressions. This discovery came about during Interview 1 while talking about $f(x + h)$; she said that the parentheses are like “a rule that is given,” (#00:29:29.4#). I asked her to clarify and she demonstrated her understanding of the parentheses below (Note: prior to this, she had written $f(x) = x^2 + 1$ and refers to this again in the conversation below):

Lian: $f(x^2)$? Let me just refer to the same one again [See written work in Figure 17 where she writes $f(x^2) = x^2 + 1$, then $(x^2)^2 + 1$ underneath it. Underneath that, she writes “$= x^4 + 1$.”] You just plug in $x^2$ into it. #00:29:54.7#

She takes $f(x^2)$ to mean that she must “plug” what’s in the parentheses, the $x^2$, into what’s on the right-hand side of the equation, in this case $x^2 + 1$. Later, she says a similar thing to what she said above, except this time in the context of $f(x + h)$. Below, I wrote $f(x + h)$ down on the paper and I asked her to discuss $f(x + h)$ in relation to $f(x)$.

Lian: . . . So the given function might be, $x + 2$ [sets this equal to the $f(x + h)$ I wrote], so now since $f(x + h)$ is telling you to plug $x + h$ every time you see an $x$. So
you want to apply that so, this \([x + h]\) is going to go into there [see Figure 18 where she draws an arrow from the \(x + h\) in \(f(x + h)\) to the \(x\) in \(x + 2\), the right side of the equal sign], so it’s \([f(x + h)]\) going to equal \(x + h + 2\). #00:32:02.3#

Similar to the previous example, in her first line, she combined the function and \(f(x+h)\). Then she took what is in the parentheses, \(x+h\), and “plugged” it into the right-hand side, \(x + 2\), to get
$x + h + 2$. She clearly has an action conception of $f(x + h)$ because to her, $f(x + h)$ is "telling" her to perform some action with $f(x)$ and she explicitly does so. However, she also says $f(x + h)$ is her result (totality). Thus, to her, $f(x + h)$ has two meanings. Shifting her conception of $f(x + h)$ as an action and totality is ideal because certain situations require conception shifting, but the notation that she uses is not.

She also cannot evaluate $f(x + h)$ unless I give her an external cue [action]:

G: Ok so what if you had—can you write this down—$f(x) = x^2 + 1$, then what would $f(x + h)$ be? #00:32:19.1#
Lian: It would just...there wouldn’t be $x + h$ since it’s not in here [she points to where $x$ is in $f(x)$]. #00:32:25.8#
G: Oh, ok. So if I said, give me this [writes $f(x + h)$]...you would say... #00:32:35.7#
Lian: Then you can plug it in. #00:32:36.7#

In this excerpt, I first asked her to tell me what $f(x + h)$ would be. Presented as $f(x) = x^2 + 1$ without anything but $x$ in the parentheses, she cannot give $f(x + h)$. Once I write $f(x + h)$, there is a procedure for her to follow. In other words, she did not view $f(x) = x^2 + 1$ as an assignment of a function with its name, but rather as a set of directions. This is another perfect example of an action conception of $f(x + h)$; she needs external cues, and the formula is a set of directions for her to follow.

On Lian’s quiz, she wrote the answer $f(x + h)^2 - x^2$ for the DQ$_h$ question (function type $g(x) = x^2$). \(^1\) I asked during Interview 2, “if you had to do this again, would this look the same?” She said, “I would probably leave $f$ over here too, put $x^2$ in parentheses” (Interview 2,

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\(^1\)Note this is not the $f(g(x))$ error, but the $g(f(x))$ error, which is discussed in the next section. It is mentioned here to give background to our conversation.
I asked her to write down what she meant and she wrote \( \frac{f(x + h)^2 - f(x^2)}{h} \). I asked her to explain each “piece” of the \( DQ_h \) and for the “piece” relevant to this section, \( f(x^2) \), she said:

Lian: ...then you plug in an \( x^2 \) into \( f(x) \). #00:34:27.2#
G: You’re plugging in \( x^2 \) into \( f(x) \)? How come? #00:34:32.5#
Lian: ‘Cuz \( x^2 \) is the solution. #00:34:35.9#

She does not give much insight into why she did that. However, her use of language is notable; she said, “plug in an \( x^2 \) into \( f(x) \).” As will be discussed later in Section 5.3, this is the exact language she uses to discuss how she got \( x^2 \) in the numerator of the \( DQ_a \).

In the next step she wrote \( \frac{(x + h)^2 - x^2}{h} \). In all the excerpts presented above, Lian had gotten the correct final answer, but for the in-between steps, she always included \( f \). More insight on these in-between steps surface in our conversation on the next error, so details are reserved for that section.

Pierre does not make this error on his quiz either, but he made it during Interview 2. He often mixed up the input, \( x \), with the output, \( f(x) \), and even said, “I don’t know how to put it into words, the difference between \( f(x) \) and \( x \),” (#00:12:02.0#). When I asked him to try, he gave a correct answer “Ok, \( f(x) \) is the whole function and \( x \) is the input to that function,” (#00:12:23.1#). But when I asked him to evaluate the \( DQ_a \) for \( \sin x \), he calls \( \sin (x) \) the input again.\(^1\)

\(^1\)In the transcription, the spoken language “sine of \( x \)” is denoted by \( \sin (x) \), whereas \( \sin x \) is used to denote the spoken language “sine \( x \)” For written work, it is transcribed as how it was written.
Pierre: Ok, so I know that the function equals \( \sin(x) \), so I plug that into the \( f(x) \) and then I subtracted \( f(a) \), any given point, and then I know that the input for this function is \( \sin(x) \) because it would, it’s almost like saying \( f(\sin(x)) \). Uh. [Writes \( f(\sin x) \), but then crosses it out]. #00:13:16.8#

I asked him how he knew \( \sin x \) was his input, hoping that he would catch his mistake and he said, “if it wasn’t the input, then \( f(x) \) wouldn’t equal \( \sin(x) \), it would equal something else,” (#00:13:32.4#). Pierre’s struggle with this seems to be different than Lian’s; Pierre explicitly refers to \( f(x) \) as an input because he’s unclear about the difference between \( f(x) \) and \( x \), while Lian misunderstood the equal sign, and used the language “plug into \( f(x) \).”

Lauri did not make this mistake on her quiz, nor did she make it in her interview. Instead she began to make the error, decided against it, and crossed it out. She says she almost did “Cuz it’s usually what you would do: you would place something into that \( x \). But, it’s not the case here.”

These four student responses show a trend in student thinking about functions and DQ evaluations. All four of the students expected something, whether it was a “solution,” a function, or an input value, to plug or place into \( x \). In each instance, students plugged the given function into \( f(x) \). Superficially, it seems Pierre does this because of his confusion with inputs and outputs. However, analyzing this error alone does not reveal the actual reason Pierre and the others are plugging the given function in the \( x \) of \( f(x) \). In particular, the denominator error analysis, in Section 5.3, is vital to understanding the reason for this error.
5.2.3  Case 3: \( g(f(x)) \) \([f(x)]^2, 1/f(x), \sin f(x)\] Error (Lian, Ray, Pierre)

Recall from Phase 1 that this error occurred most frequently for the function type \( x^2 \). This discovery along with this error’s relationship to the previous error were the main motivations for investigating this error. Lian and Ray, who made this error on their quizzes, were not among the students who received a 0 on this part of the question since the correct answer was also present on the quiz. Pierre, however, wrote \( f(x+\Delta x)^2 \), for evaluating the DQ\(_h\) for \( x^2 \). When asked about it, he fixed his mistake and knew that it was wrong.

As shown in Lian’s work from the previous subsection, she wrote \( \frac{f(x+\Delta x)^2 - x^2}{h} \) on her quiz for DQ\(_h\) evaluation. When asked to explain what she did, she began to explain the first “piece”, \( f(x+\Delta x)^2 \), “When you set up the difference quotient you plug in \( x+\Delta x \) for all, or, yeah for all \( x \)'s. So, I did that only for the first part 'cuz that’s what the rule is saying” (#00:32:13.2#). I asked about the \( f \) in particular (Note: as discussed in the previous error category, during our interview she writes \( (x+\Delta x)^2 - \frac{f(x+\Delta x)^2}{h} \) after \( f(x+\Delta x)^2 - \frac{f(x^2)}{h} \)):

Lian: [6 second pause] I’m just showing that I plug in \( x+\Delta x \), ’cuz I take it out over here in the next step. #00:32:30.8#
G: Yeah what happens to that \( f \)? #00:32:29.7#
Lian: I’m just doing step by step. [?] #00:32:36.3#
G: Ok so what does that mean, ‘\( f(x+\Delta x)^2 \) squared’? #00:32:43.0#
Lian: What does it mean? It means that you plugged it \( x+\Delta x \) in the \( x^2 \). #00:32:46.0#

Her logic does not completely follow. However, her use of language tells us that the presence of \( f \) is some intermediate step between \( f(x+\Delta x) \) and \( (x+\Delta x)^2 \). She said she is “showing” what she did, four times throughout our conversation. In a situation where just the answer was required, she would write down the correct answer. However, when she wanted to show her work, she
included the work with the $f$’s. Recall she did this in the previous error category as well. When shown the student sample with the same mistake, she offers that “maybe he’s [sic] just showing his steps” (Interview 2, #00:33:12.2#).

A little later, I asked her what $f(x + h)^2$ meant again and she responded:

Lian: [13 second pause] I don’t know. I kinda remember our teacher saying, ‘oh just plug it in, just plug in $x + h$’ and then he just got rid of the $f$ [laughs]. #00:37:54.5#

G: Oh so ok, there is an $f$ there, and in the next step you get rid of it? #00:38:01.5#

Lian: I don’t know why he gets rid of it though. #00:38:06.3#

G: Tell me why it is there in the first place. #00:38:08.4#

Lian: [8 second pause] Because that’s the difference quotient? #00:38:21.7#

In this excerpt, we can start to see the influence of the interpretation of the instructor’s language on evaluating the $DQ_h$. This surfaces in my conversation with Ray as well.

Ray had the correct answer for $DQ_h$, but like Lian, it was accompanied by the incorrect work: $\frac{f(x + h)^2 - f(x)^2}{h}$. Unlike Lian, Ray was consistent with the type of mistake he made on this question. He performed the same error for $f(x + h)$ and $f(x)$, whereas Lian had made one category of mistake with $f(x + h)$ and another category with $f(x)$. In Interview 2, when I asked him about this work, he corrected his mistake with no prompt from me. I asked him about the $f$’s in his original work, he paused for 6 seconds and then said he “just assumed you put it there” (#00:23:51.0#) because:

Ray: It wasn’t explained to us at the beginning what all this stuff really meant and then our professor went and explained, even such things as the limits—let’s say as $x$ goes to 0 $\left[\lim_{x \to 0}\right]$. It was later explained to us, you have to have this $\left[\lim_{x \to 0}\right]$ in front of each step that you do it until the very last step. So little things
like that. I don’t really know what’s going on, I just did it for the sake of doing it. #00:24:25.8#

This lack of understanding of the limit supports the $f(x)^2$ error. For Ray, he just thinks having the $f$’s in the step before the final step is another example of writing something in front of your work, even if you do not understand why. He does not view the extra $f$’s as problematic, because he relates it to his lack of understanding the purpose of writing $\lim_{x \to 0}$. This excerpt, along with the statement he made regarding the previous error: the “TA just told us to plug them in, that’s what I did,” shows the impact of instructor language on student work.

This case study showed that Lian and Ray both had these errors on their quizzes followed by the correct answer. Lian indicated that this was because she was “showing her work,” while Ray just said that he just followed the TA’s orders. Lian also expressed some confusions that came from her interpretation of her instructors’ written or spoken actions. Lian and Ray both tried to make sense of the directions that have been given to them, “plug it in.” In this instruction, there are no details of what to plug into where, so they choose something to plug (the expression), and then choose a place to plug it into (the $x$). Even though they both correctly evaluated the DQ’s in the end, the extra step of including the $f$’s is their attempt to reconcile the instructions given to them with their own understandings of how to evaluate the DQ’s. This fundamental confusion of the “plug it in” directions is discussed fully in the Intercase Analysis section (5.3).

5.2.4 Case 4: Denominator Error $[x^2 - a, 1/x - a, \sin x - a]$ (Lian, Pierre, Lauri, Sterne)

Student thinking for this error are related to student thinking for the previously discussed errors. That is, the desire to plug in, and weak understandings of what constitutes an input or
output. In fact, all but one (Sterne) of the entire sample of students who made the denominator error on the quiz, also made one of the previous three errors. This was not true the other way around. That is, of the students who made one of the previous three errors, most did not make the denominator error.

Of the students who are categorized in this error, Pierre and Sterne actually made this mistake on the quiz; the others performed it during the interview. Pierre’s language is similar to his language shown in Section 5.2.2. That is, he is unclear about what to consider an input or output. Below I asked him to explain how he evaluated $DQ_a$ for $f(x) = \frac{1}{x}$ on his quiz.

Pierre: I just—because I know my $f(x)$ is $\frac{1}{x}$, and I—if I’m recalling it correctly, it’s $f(x) - f(a)$ over, whatever your value for $f(x)$ is minus $a$ [writes $\frac{f(x) - f(a)}{f(x) - a}$]. It’s, it’s just $x - a$ [crosses out $f(x)$ in the denominator and writes $x$]. #00:09:40.5#

After he talked about how he cannot evaluate $f(a)$ [discussed previously in Section 5.2.1], I asked him to talk about the denominator. Even though he corrected his mistake above, in the excerpt below, he’s still fixated on the fact that the $x$ is the $\frac{1}{x}$.

Pierre: I mean I just also plugged in $\frac{1}{x}$. I know what I did; I just don’t know how to describe it. Let me think, how’s a good way to say this? Because I know my $x$ is $\frac{1}{x}$, because if $f(x) = \frac{1}{x}$, that’s like saying... how do I say what I’m trying to say? Basically, I just plugged and chugged. ’Cuz I know that $f(x) = \frac{1}{x}$, so I just plugged it into this general formula. #00:11:07.2#

While evaluating the denominator of the $DQ_a$ for $g(x) = \sin x$, Pierre said, “Basically my input is $\sin(x)$ so I plug that in for $x$ and [then] minus $a$” (Interview 2, #00:13:16.8#).
this statement, it seems to Pierre that a consequence of being an input is that it gets plugged into \( x \).

Calling the function itself the input is language that is not restricted to his own work. While discussing an incorrect evaluation of \( \frac{1}{x} \) (Sample 1; Appendix D.1), who made the same denominator mistake as he did, he commented:

Pierre: The rest of the problem is fine ’cuz they just subtracted \( f(a) \) and then your actual \( x \)-value equals \( \frac{1}{x} \). [?] So the rest is fine. #00:16:41.6#

G: Ok. Great. How do you know the actual \( x \)-value is \( \frac{1}{x} \)? #00:16:52.4#

Pierre: I guess I just assumed it from \( f(x) = \frac{1}{x} \). #00:16:56.8#

So far in Pierre’s language, he has called \( \sin(x) \) an input, and has also said that \( \frac{1}{x} \) and \( x \) are equal, but both \( \sin(x) \) and \( \frac{1}{x} \) are outputs of \( f(x) \). Superficially, it seems like he’s mixing up inputs and outputs. If this is the case, then it is easy to write \( \frac{1}{x} \) for \( x \). However, a deeper analysis provided in the next section will show that Pierre is clear on what an input is.

Sterne also made this denominator mistake on her quiz. When asked to redo the \( \frac{1}{x} \) problem for \( x^2 \) during Interview 2, she said:

Sterne:...You know \( f(x) \), this is \( f(x) = x \) [sic], so you would plug that in for that [points to \( f(x) \) in the difference quotient] and then \( f(a) \), whatever the \( a \)-value would, you would just square the \( a \)-value, ’cuz that’s squared. And then, wait a minute. [30 second pause] Oh. Ok sorry I got confused. Ok so, the—you keep the \( x \)-value squared here [the denominator] and you don’t keep the \( a \)-value squared because it’s not \( f(a) \), so you just leave \( a \). [Student has written \( \frac{x^2-a^2}{x^2-a} \)].#00:12:31.8#

Notice how long it took her to think about the denominator. After thinking about it, she still did not give the correct denominator. However, a little bit later she said, “I almost think that
it should just be $x$, like looking back on this” (Interview 2, #00:12:59.3#) and continued, “I should have just left this as $x$ which is what’s in the equation. But I’m not sure which way is right” (Interview 2, #00:14:03.5#). Throughout our discussion, Sterne could not decide which way was correct, even when shown the student sample with the same error. In addition to her argument for $x - a$ as the denominator, she justifies $x^2 - a$ with the reason “the $x$-value should equal $x^2$” (Interview 2, #00:14:03.5#). Note that this is similar to how Pierre justified this error for $\frac{1}{x}$. Lauri expresses a similar sentiment.

Lauri did not make this error on her quiz, but she almost made it on the interview. When I asked her why, she said that she thought $f(x)$ and $x$ were the same thing. When she was asked to comment on a student’s evaluation of the DQ$_a$ for $\frac{1}{x}$ who had $\frac{1}{x} - a$ as a denominator, she wrote “$x = \frac{1}{x}$” and said, “So they assumed that $x$ is equal one $x$ [sic]” (Interview 2, #00:15:15.6#).

Recall from the discussion on the $f(a)$ error in Section 5.2.1, that Lian used the DQ$_h$ instead of the DQ$_a$ to answer Question 1. When I asked her to reevaluate DQ$_a$ during Interview 2, Lian echoes Sterne and Lauri by saying $x$ is $x^2$. Below is our discussion.

Lian: Over the, denominator you have $x - a$, so $x$ is $x^2$, minus $a$. #00:17:21.1#
G: Ok, how come? #00:17:26.8#
Lian: 'Cuz. [Laughs. Eleven seconds pass.] You would just leave it as $x - a$. #00:17:47.9#
G: Tell me what you’re thinking. #00:17:57.1#
Lian: You would just leave it as $x - a$. So this is—[crosses out the denominator and rewrites $\frac{x^2 - a^2}{x - a}$] so these would be constant or numbers. #00:18:05.8#

When asked why she first made this mistake, she says, “‘Cuz, if you’re given $x$, it’s usually like a number or another equation, so you just plug that in. And the same thing with $a$” (Interview
2, #00:19:00.0#). Recall that this is similar to what Lauri said regarding the \( f(g(x)) \) error discussed previously in Section 5.2.2.

In this case analysis, the obsession with plugging in, surfaces again. Students also have difficulty with understanding the difference between \( f(x) \) and \( x \), and frequently say \( x \) is \( x^2 \), or \( x \) is \( \frac{1}{x} \), or that the input is \( \sin x \). This difficulty is discussed in detail in the next section on the inter-case findings.

5.3 Findings from Inter-Case Analysis

Table XXVIII gives a quick summary of how many interviewees performed each of the errors investigated in this case study. Case 1, no \( f(a) \) evaluation error, was the most prominent error among Phase 2 participants. The table also shows students performing one or more of the errors, namely, Lian, Pierre, Ray, and Lauri. This section will review all the findings from the previous cases and perform a cross case analysis that will draw patterns in the data.

For no \( f(a) \) evaluation, some students expected \( f(a) \) to be given in order to properly fill in the DQ\( a \). These students did not see a connection between \( f(x) \) and \( f(a) \). For example, if there was an error with \( f(x) \) in the DQ\( a \), there was not necessarily an error with \( f(a) \), or vice versa. Drew, who evaluated \( f(a) \) correctly, explained this phenomenon when she was shown error samples from other students’ quizzes in Interview 2. When she was asked to discuss Sample 1 (Figure 19; also Appendix D.1), she said:

Drew: ...Well, they didn’t do the same thing to both the \( f(x) \) and the \( f(a) \). They left the \( x^2 \) for the \( x \). But they didn’t do that to—even though the \( x \) and the \( a \) are really—they’re not the same variable, it’s the same idea; they’re both variables, so you should treat them the same way. #00:26:09.6#

G: Can you talk more about that? #00:26:11.8#
TABLE XXVIII: ERROR CATEGORIES IN WHICH STUDENTS ARE MEMBERS (PERFORMED EITHER ON QUIZ OR IN INTERVIEW)

<table>
<thead>
<tr>
<th>Student</th>
<th>1. No $f(a)$</th>
<th>2. $f(g(x))$</th>
<th>3. $g(f(x))$</th>
<th>4. Denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>Ray (A)</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>•</td>
<td>•</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>Drew (D)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Turner (C)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jo (A)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nat (–)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sterne (F)</td>
<td></td>
<td></td>
<td>•</td>
<td></td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>•</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>4</strong></td>
<td><strong>3.5</strong></td>
<td><strong>2</strong></td>
<td><strong>3.5</strong></td>
</tr>
</tbody>
</table>

\(^a\) Denotes that the student did not perform this error on the quiz nor during the interview, but instead s/he wrote it, and then crossed it out.
Drew: If you have 2 variables, even though they’re not the same thing, you should just, I don’t know, treat them the same way. I don’t understand why they, did $x^2$ for the first one and not $a^2$ for the second one. #00:26:38.8#

In this way, students who did not evaluate $f(a)$ viewed it as a separate concept from $f(x)$.

However, when given the external cue of asking some students what is $f(a)$ when $a$ was a specific number, they were able to move to an action conception and evaluate $f(a)$. Regarding APOS, the students who could not evaluate $f(a)$ had different stages of conceptions of $f(a)$ than of $f(x)$, even though understanding of the former is a subset of understanding of the latter. This mismatch suggests that these students have not reached a full conception of the function concept.
Analysis of the second error type, e.g., \( f(x^2) \), hints at the effect of instructors’ language on this error. In addition, the students’ own spoken language regarding this error was also revealing. Most often, they stated that they were plugging into the \( f(x) \) or into the \( x \) of the \( f(x) \). There was also evidence that functions themselves were perceived as inputs. This perception is appropriate as long it is specified what they are inputs of, e.g., \( f(x) \) is the input for \( g(f(x)) \).

The findings from the third case, e.g., \( f(x)^2 \), reinforce the influence of instructors’ spoken and written actions that first surfaced from the previous case. These errors seem to be a result of understanding how the instructors’ actions relate to the students’ prior knowledge about functions.

Analyzing the students’ thinking of the denominator errors complemented the findings from the second error to show a well-rounded picture of students’ understanding of DQ evaluation. In the discussions about the denominator error, students frequently said that \( x \) is the given \( f(x) \). Data also suggested that \( x \) was viewed dually as the variable itself and as an input, which is true with respect to \( f(x) \). However, in situations such as function composition, e.g., \( f(g(x)) \), \( x \) is only the set of possible inputs of \( g \), and not of \( f \). As such, care needs to be taken in choosing appropriate times for \( x \) to represent the input.

This intercase analysis showed that students have a tendency to plug the given function into the \( f(x) \)’s of the DQ’s, or plug the \( f(x) \) of the DQ’s into the given functions. The findings across cases also shows that students referred to \( f(x) \) as an input, or called \( x, f(x) \). There was also
evidence in more than one case that showed students struggling to make sense of instructors’
written or spoken language.

5.4 Conclusion

The remainder of this chapter will use these findings from across the four cases to reveal
students’ thinking on functions and the evaluations of the DQ’s. Below is a summary of students’
original statements that are representative of student responses from the previous sections.

- “Basically my input is sin \(x\), so I plugged that in for \(x\)”
- “\(x\) is \(x^2\)”
- “your actual \(x\)-value equals \(\frac{1}{x}\)”
- “I thought these two were the same [underlines \(f(x)\) and \(x\)]”

As depicted in these statements, students called the given \(f(x)\) \((x^2, \frac{1}{x}, \text{or sin} x)\) an input, and
said \(x\) equaled the given \(f(x)\). In this section, I will suggest why students were making these
statements and how these comments tended to result in plugging the given function \((x^2, \frac{1}{x}, \text{or sin} x)\)
into the \(x\) of \(f(x)\) in the DQ and also into the \(x\) in the denominator of the DQ. Lastly, I
will make suggestions towards why students did not see a relationship between \(f(x)\) and \(f(a)\).

An examination of the language students used while discussing their evaluations of the DQ’s
showed that the language fell into one of two categories, which are represented by the following:

Jo: I just used the \(f(x)\) function that they gave me and just plugged it into the equation
that they gave me, the difference quotient. [Writes \(\frac{x^2 - a^2}{x - a}\).]
Pierre: I know that the function equals \( \sin(x) \), so I plug that into the \( f(x) \) and then I subtracted \( f(a) \), any given point, and then I know that the input for this function is \( \sin(x) \) because it would, it’s almost like saying \( f(\sin(x)) \).

In the first example, Jo says to plug \( x^2 \) into the equation, the difference quotient, as opposed to Pierre who says to plug \( \sin x \) into \( f(x) \). Both of them use the word plug, but the difference is that Jo correctly plugs into the DQ, and Pierre mistakenly plugs into the function.

Pierre, Lian, and Lauri were three of the four students who were categorized into the \( f(g(x)) \) error category and who also used the language “plug into the \( f(x) \)”\(^1\). Although Lian evaluated the \( f(x) \) portion correctly in the DQ, she used the same language to get \( f(x^2) \) for the \( f(x) \) in the DQ\(_h\). Recall that Lauri did not fully carry out this mistake, but like Lian, she used this language to describe DQ\(_h\) evaluation.

Students from both language categories often referred to the DQ’s as equations, formulas, or functions. For instance, Lauri evaluated the DQ and said, “The formula’s [writes \( \frac{f(x) - f(a)}{x - a} \)—so you’re taking \( \frac{1}{x} \) and plugging into the formula” (Interview 2, #00:10:57.8#). As such, students are looking to plug the given functions into where \( f(x) \) is in the DQ\(_h\). Notationally speaking, students are thinking like this:

\[
DQ_a(f(x)) = \frac{f(x) - f(a)}{x - a}.
\] (5.1)

\(^1\)For a detailed look at the exact language the students used, see Table XLIII, and Table XLIV in Appendix F.
In this representation, the DQa is viewed as a procedure, or action, and the students are expecting \( f(x) \) as an input. Thus, this could explain why some of the students called \( f(x) \) an input; although they did not specify of what \( f(x) \) is an input, evidence suggests that they were thinking of it as an input of DQa. Viewing the DQa as Equation 5.1 suggests that students were not confusing the definition of input with output as originally thought.\(^1\)

Considering the DQ as in Equation 5.1 does not guarantee an incorrect evaluation, as demonstrated by Jo’s use of language. However, if in addition to failing to mention of what \( f(x) \) is an input, this conception is accompanied by a weak process view of inputs, i.e., if students were not able to coordinate \( x \) as an input for \( f(x) \), while attending to \( f(x) \) as an input for DQa, then students may have difficulties with evaluating the DQ as represented by the above formula. In addition, students who do not have a strong process conception of \( x \), i.e., cannot coordinate \( x \) as an input for \( f(x) \) while attending to the notion of \( x \) as a variable, may also have trouble with DQ evaluation. These two areas of weakness in understanding \( x \) and “input” may cause students to use them interchangeably. Thus, if \( f(x) \) is viewed as an input as in Equation 5.1, while \( x \) is also viewed as an input, then the statements such as “\( x \) is \( x^2 \)” follow naturally since they are both inputs.

If \( x^2 \), \( \frac{1}{x} \), or \( \sin x \) are conceived as inputs of the DQ, along with the incorrect notion that \( x \) is \( x^2 \), \( \frac{1}{x} \), or \( \sin x \), it is instinctual to plug those into the \( x \) of the \( f(x) \). Not only into the \( x \) of the \( f(x) \), but students will also have a tendency to plug the given functions into the \( x \) of the

\(^1\)During the interviews, it was assumed that the word input was used to refer to the input of functions, or of \( f(x) \), because those were the only things in our discussions that had inputs.
denominator of $DQ_a$. In fact, Pierre, Lian, and Lauri, and who were categorized into the $f(x^2)$, $f(\frac{1}{x})$, $f(\sin x)$ error category, were also categorized into the denominator error category.

A reevaluation of the confounding student statements from above can show that students used the word “$x$” incorrectly to represent the word “input”, but even if they used the word “input” correctly, they did not specify for which expressions these are inputs. By adding what expressions $f(x)$ is an input for and also replacing the word “$x$” for “input” the incorrect statements from above can be transformed into the language similar to Jo’s. Below, the additions are in italics and replacements are denoted by the strikethrough.

- “Basically my input of the $DQ$ is $\sin(x)$, so I plugged that in for $x$ the input of the $DQ$, which is $f(x)$”
- “$x$ the input of the $DQ$ is $x^2$”
- “your actual $x$-value input of the $DQ$ equals $\frac{1}{x}$”
- “I thought these two were the same \( f(x) \) and \( x \) because $f(x)$ is the input of $DQ$ and $x$ is the input of $f(x)$.”

These distinctions in language could help move students' from using language like Pierre’s to language like Jo’s, with the ultimate outcome of correct $DQ$ evaluation.

Additional analysis of how students discussed $f(a)$ showed that students expected it to be given along with $f(x)$. Therefore Equation 5.1 could be adjusted to reflect this student thinking:

$$DQ_a(f(x), f(a)) = \frac{f(x) - f(a)}{x - a}.$$  (5.2)
Thinking about the DQ’s in this way suggests that \( f(x) \) and \( f(a) \) are two different entities and that led students to think they did not have enough information to evaluate \( f(a) \).

This new adjustment does not change the fact that \( f(x) \) can be considered as an input to the DQ\(_a\). However, if students have underdeveloped understandings of the concepts input and \( x \), students may make the \( f(x^2) \) error, or the denominator error.
CHAPTER 6

DEVELOPMENT OF DATA ANALYSIS TOOLS FOR APOS

The purpose of this chapter is to understand some of the difficulties with using APOS theory to assess student understanding and to suggest types of data analysis tools that will encompass the following:

- raw data examples, to give researchers an idea of how data was categorized and how to categorize their own data, and
- strength of levels for theory conceptions, to distinguish students within the same conception category.

This way, future researchers get a sense of what constitutes an action, process, totality, or object conception of function, composition, and $f(x + h)$. Furthermore, researchers can see the progression and hierarchy of APOS conceptions through the words of students.

Below is a review of the general definitions used in this chapter.

Action. This is the first stage students enter to begin their understanding of a mathematical concept. Here, students explicitly transform objects through a step-by-step procedure.

Process. This is the stage has two requirements. The first requirement states that students must imagine the transformation. There are at least two ways they
can do this. Students can treat transformations step-by-step as they did in the previous conception category, but they do not need to explicitly perform the transformation; or they can treat transformations more generally. The second requirement states no external stimuli were needed to trigger transformations. Though it is not a requirement to show ability to reverse the steps of the transformation, doing so is a good indication of having a process conception. Another indication is ability to combine the transformation with other processes.

Totality. At this stage, students recognize that the transformation produces a result, or entity, but the student does not explicitly act on it.

Object. This stage has two levels. Students must realize they can act on the mathematical concept as a totality and then they must explicitly do it to be considered a member of the object conception category.

These general definitions of each stage and their applications to the concepts of function, composition, and $f(x + h)$ were matched to each other. Examples of language used by students from this study, who demonstrated membership in that respective stage, were then matched to the concept-specific definitions. This process was used to create analysis tables. Where there was inadequate information in the available research, definitions were created based on my data. For example, since the totality stage is still in its infancy in the literature, the only definition applied to just infinite repeating decimals. Thus, the above general definition is based on my
interview data and the use of the stage by the researchers who coined totality (Dubinsky et al., in press).

The data analysis tables were then used to create assessment tools to classify students’ conceptions of function, composition, and $f(x + h)$. These assessment tools also show the strength levels of conceptions. They are not meant to draw any generalizations about the students from this study; the sample is too small to do this. For a larger group of students, this may be useful to evaluate the effectiveness of a pre-calculus or calculus course. For instance, the assessment tools could be used to categorize students on a pre-test and then again on a post-test to see if there was a change in their conceptions due to a particular instructional treatment. However, for the purposes of this study, these assessment tools serve as models for researchers wishing to develop similar tools to apply APOS to student thinking on other mathematical topics.

The resulting data analysis tools for composition, $f(x + h)$ evaluation, and functions are explained in the following sections. The data analysis table for function is presented last, rather than first, because aspects of composition and $f(x + h)$ evaluation are included in that table.

6.1 Function Composition

Table XXIX connects general definitions of APOS to topic-specific definitions for function composition, and then to examples drawn from interview data gathered from this study. For example, the first row shows that the action definition of function composition requires explicitly replacing or substituting every occurrence of one variable in an expression with a second expression. In other words, students who explicitly plugged in an expression into a variable were
classified as having an action conception. This is demonstrated in the last column with Pierre’s explanation of function composition.
### TABLE XXIX: APOS THEORY FOR FUNCTION COMPOSITION

<table>
<thead>
<tr>
<th>Stage</th>
<th>General Definition</th>
<th>Definition Applied to Composition</th>
<th>Example from Student</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Action</td>
<td>“set of step-by-step instructions performed explicitly to transform physical or mental objects” (Dubinsky et al., in press)</td>
<td>procedural task in which every occurrence of one variable in an expression is explicitly replaced/substituted with another expression (Breidenbach et al., 1992; Oehrtman et al., 2008)</td>
<td>“if you have $f(x) = x^2$ and $g(x) = x + 1$. You can say $f(g(x))$ would be—you’d plug your $g(x)$ into your $x$, in the $f(x)$. So it would be $(x + 1)^2$—would be your $f(g(x))$. ” -Pierre</td>
<td></td>
</tr>
<tr>
<td>Process</td>
<td>• procedure is entirely imagined</td>
<td>• can imagine the composition (procedurally same as action)</td>
<td>• “plug in a specific number into $g(x)$ first and then find—solve for what $g(x)$ would be for that, and then plug that solution into the function $f(x)$.” -Turner</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• external stimuli not required</td>
<td>• “coordination of two input-output processes; input is processed by one function and its output is processed by a second function.” (Oehrtman et al., 2008, p. 159)</td>
<td>• “The $g$ values [would be inputs for $f$, because the] domain would be the $x$-values that you’re inputting in. [S]o then, you’d first look at $g$ because that’s what you’re inputting into $f$.” -Sterne</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+ can be reversed (Asiala et al., 1996; Breidenbach et al., 1992)³</td>
<td>• does not need $f(g(x))$ or $f \circ g(x)$ cue</td>
<td>• NA</td>
<td>+</td>
</tr>
<tr>
<td>Totality</td>
<td>recognition of concept as an entity, but does not act on it</td>
<td>recognition that the final result of a composition is a function or a value of a function</td>
<td>“When you compose different functions to make a function.” -Hudson</td>
<td></td>
</tr>
</tbody>
</table>

³“+” denotes an indication of a process for in data analysis table and all subsequent data analysis tables.
The next stage in the table reflects the two requirements of a process conception. In the first requirement, the composition is entirely internal to the student; that is, it is imagined. There are two dimensions to this internal understanding. The first is acknowledgement that the composition is procedurally the same as an action, but the second is a deeper understanding that requires coordinating two functions as processes. Sterne demonstrates this higher level of understanding in the last column, compared with Turner’s answer. The second requirement does not have a student example because it is not possible to show exact quotes of students not needing external cues from me; they simply just did not need them. Similarly, for the last row (which is a sufficient indication as opposed to a necessary requirement), it is just a measure of time, so there is no student quote applicable.

Note that Table XXIX ends at the totality conception and does not cover an object conception of function composition. This is because after composing two functions to get a new function or value, it is not necessary to apply an action or process to that new result. Thus, the new result remains a totality. Any action or process taken on the new function would be an indication that the student has an object conception of function rather than of function composition.

The format of discussion has been focused on the lateral progression of the table, i.e., general theory connected to its application to composition connected to actual student data. However, it is also helpful to read the last column of the table from top to bottom to see the progression of the stages through the students’ words. The difference in the stages becomes clear through these examples.
TABLE XXX: ASSESSMENT OF THE STRENGTH OF STUDENTS’ PROCESS CONCEPTION OF FUNCTION COMPOSITION

<table>
<thead>
<tr>
<th>Student</th>
<th>Imagine</th>
<th>Coordinate</th>
<th>No External Cue</th>
<th>Strength</th>
<th>Reverse (≤ 1 minute)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>✓</td>
<td>✓</td>
<td>1</td>
<td></td>
<td>(5'57&quot;)</td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>✓</td>
<td>✓</td>
<td>1</td>
<td></td>
<td>(1'04&quot;)</td>
</tr>
<tr>
<td>Ray (A)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+</td>
</tr>
<tr>
<td>Lauri (C)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+</td>
</tr>
<tr>
<td>Drew (D)</td>
<td>✓</td>
<td>✓</td>
<td>2</td>
<td></td>
<td>+</td>
</tr>
<tr>
<td>Turner (C)</td>
<td>✓</td>
<td>✓</td>
<td>2</td>
<td></td>
<td>+</td>
</tr>
<tr>
<td>Jo (A)</td>
<td></td>
<td>✓</td>
<td>1</td>
<td></td>
<td>+</td>
</tr>
<tr>
<td>Nat (−)</td>
<td>✓</td>
<td>✓</td>
<td>2</td>
<td></td>
<td>(9'34&quot;)</td>
</tr>
<tr>
<td>Sterne (F)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>3</td>
<td>+</td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>✓</td>
<td></td>
<td>1</td>
<td></td>
<td>+</td>
</tr>
</tbody>
</table>

The next step is to use Table XXIX to create an assessment tool that shows the strength of students’ process conception. Table XXX was created with two dimensions from the first requirement, the second requirement, and the indication of reversal (since students had the opportunity to exhibit their ability to reverse the composition).

Notice even though Ray and Lauri have no observable requirements, they both indicated that they possessed a process conception through their ability to reverse a composition in under one minute. Therefore, all the students in this study displayed a process conception of function composition.

1 The action and totality stages do not require strength levels because they do not have multiple levels like process.
composition. However, it is apparent that some students had a stronger process conception than others. For example, Sterne had the strongest process conception of function composition. This observation is made possible by this new dissection of the process conception. Before this study, all these students would have been classified into a process conception of function composition. These distinctions of the strength of the conception could account for differences in student explanations found during their interviews.

After looking at the strengths of the students’ process conceptions of function composition, the student interviews were analyzed for evidence of action and totality conceptions of composition. Results are shown in Table XXXI. Notice that only three students displayed a totality conception of composition. In these three cases, the participants acknowledged that the output of a composition was another function.

6.2 $f(x + h)$ data table development

The same course of action was taken to analyze students’ conceptions of $f(x + h)$. That is, topic-specific definitions from the literature that matched the general definitions were gathered and matched with raw data examples. When topic-specific definitions were not found in the literature, they were derived from the general definitions.

Table XXXII, which describes APOS for $f(x + h)$, follows a similar format to the data analysis table for composition. However, during data analysis, there was evidence for additional stages not provided by APOS. The first stage added was the pre-action stage, where students showed misconceptions, in this case of $f(x + h)$ understanding. In addition, students seem to make general statements about $f(x + h)$, signaling a process conception, but on the other hand,
TABLE XXXI: ASSESSMENT OF ACTION, PROCESS, TOTALITY CONCEPTION OF FUNCTION COMPOSITION

<table>
<thead>
<tr>
<th>Student</th>
<th>Action</th>
<th>Process</th>
<th>Totality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Ray (A)</td>
<td>1</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>1</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>Drew (D)</td>
<td></td>
<td>2+</td>
<td></td>
</tr>
<tr>
<td>Turner (C)</td>
<td>1</td>
<td>2+</td>
<td></td>
</tr>
<tr>
<td>Jo (A)</td>
<td>1</td>
<td>1+</td>
<td></td>
</tr>
<tr>
<td>Nat (–)</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Sterne (F)</td>
<td></td>
<td>3+</td>
<td>1</td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>1+</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

\textsuperscript{a} Blank spaces indicate lack of observable data to show evidence of possessing this conception, rather than the absence of this conception.

these statements were also restricted, evidence of an action conception. This observation was the basis of an intermediate level between action and process.
<table>
<thead>
<tr>
<th>Stage</th>
<th>General Definition</th>
<th>Definition Applied to $f(x + h)$</th>
<th>Example from Student</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-action</td>
<td>incorrect statements about concept</td>
<td>• $f(x + h)$ and $f(x)$ are unrelated</td>
<td>• “it’s just a different function ... this is asking you in relation to $x$. But this is asking you in relation to $x + h$.” -Hudson</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• solve for $x$ or $h$</td>
<td>• “say $h$ wasn’t given to you, I think you just solve it ... there would remain an $h$ in the equation.” –Lauri</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• $h$ is added to the “equation”</td>
<td>• “you’re adding whatever $h$ value would be to the $g(x)$.” –Sterne</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• vertical transformation; $f(x + h)$ is an up or down shift of $f(x)$</td>
<td>• “cos $(x + h)$ would be... above it... you’re adding a value to it. It would make it higher” –Drew</td>
</tr>
<tr>
<td>Action</td>
<td>“set of step-by-step instructions performed explicitly to transform physical or mental objects” (Dubinsky et al., in press)</td>
<td>• explicitly evaluates $f(x + h)$ with numbers</td>
<td>• “you would have $f(2)$ and the function is $x^2 = 2^2$. Let’s say $h = 3$, you are gonna have $f(2 + 3) = f(5)$, then you take the square, that equals 25.” –Ray</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• procedural task in which every occurrence of $x$ in a specific $f(x)$ is explicitly plugged into/replaced/substituted with $x + h$</td>
<td>• “I’m going to use $3x + 5$. [Writes $f(x + h) = 3(x + h) + 5$]” –Nat</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• external prompt needed</td>
<td>• See Lian’s Interview 1 from #00:32.19.1# to #00:32.36.7#</td>
</tr>
</tbody>
</table>

Continued on next page
<table>
<thead>
<tr>
<th>Stage</th>
<th>General Definition</th>
<th>Definition Applied to $f(x + h)$</th>
<th>Example from Student</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action/Process</td>
<td>Evidence of generality, but not a full process, e.g. exhibits some restriction</td>
<td>• has at least a process view $x + h$, but mention of output missing</td>
<td>“$x$ is all real numbers, and then $h$ is a variable or like, it’s not a constant, well it can…” Jo</td>
</tr>
<tr>
<td>Process</td>
<td>• procedure is entirely imagined</td>
<td>• can imagine the $f(x+h)$ evaluation (procedurally same as action)</td>
<td>“plug in $x+h$ to every $x$ that you see in the given function” Lian</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• mentally runs through continuum of points $x+h$ for $f(x)$ while attending to resulting impact on the output</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• external stimuli not required</td>
<td>“you could have $h$ be a constant, or it can, like this $[x]$ can be your input and if you add it to another number it can—it can totally change things.” Pierre</td>
</tr>
<tr>
<td></td>
<td>• does not need cue</td>
<td></td>
<td>NA</td>
</tr>
<tr>
<td>Totality</td>
<td>recognition of concept as an entity, but does not act on it</td>
<td>• $f(x+h)$ is a result of a horizontal shift of $f(x)$</td>
<td>“shifts it. Either, I think to the right, or something.” Lauri</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• recognition that $f(x+h)$ is a an output of $x+h$</td>
<td>“$f(x+h)$ equals this $y$—intercept [the $y$—value that $x+h$ is mapped to on graph of a line that he drew].” Ray</td>
</tr>
</tbody>
</table>
The process conception of $f(x + h)$ has two levels for the first requirement, similar to the process conception of function composition. That is, the first level is the same as an action except that students can imagine $f(x + h)$ without explicitly doing it, but they still describe it in the procedural sense: plug $x + h$ into every $x$. The second level requires students to imagine the continuum of input values with respect to the corresponding output. Similar to function composition, $f(x + h)$ ends at the totality conception. This is because students are asked to discuss $f(x + h)$, but not asked to act on it.

After this data analysis table was created, it was used to create tools to assess students’ conceptions of $f(x + h)$. The results from analyzing the strengths can be found in Appendix F. Overall student conceptions can be found in Table XXXIII.\(^1\) The first column represents the strength of the pre-action conception for $f(x + h)$. Recall that in this stage, incorrect statements about $f(x + h)$ are made; the gravest misconception being “$f(x + h)$ and $f(x)$ are unrelated”. Students got a $-1$ for each of the number of categories of the pre-action conception (shown in Table XXXII) into which the students’ responses fell, i.e., not the total number of inaccurate statements during the $f(x + h)$ discussion.\(^2\) Only the students who received an A or B in the course did not show evidence of any misconceptions regarding $f(x + h)$.

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\(^1\)This analysis only covers students’ responses to the question about the relationship between $f(x + h)$ and $f(x)$. It does not reflect all conversations where $f(x + h)$ was mentioned.

\(^2\)For a detailed account of which categories a particular student’s response fell into, see Table XXXVI in Appendix F.
TABLE XXXIII: ASSESSMENT OF ACTION, PROCESS, TOTALITY CONCEPTION OF $f(x + h)$

<table>
<thead>
<tr>
<th>Student</th>
<th>Pre-action</th>
<th>Action</th>
<th>A/P</th>
<th>Process</th>
<th>Totality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>−1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>0$^a$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Ray (A)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>−2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Drew (D)</td>
<td>−2</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Turner (C)</td>
<td>−2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Jo (A)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Nat (–)</td>
<td>−2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Sterne (F)</td>
<td>−2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>−1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$^a$ Zero denotes the absence of incorrect statements about $f(x + h)$, whereas, the blank spaces indicate lack of observable data to show evidence of possessing this conception, rather than the absence of this conception.

The strength levels of the remaining columns were determined the same way. Hudson was the only person who did not display any evidence of possessing a process conception of $f(x + h)$. In fact, in her responses she stated three times that $f(x+h)$ and $f(x)$ are two different, unrelated functions. Three others were able to conceive of $f(x + h)$ as totalities (either as a horizontal shift of $f(x)$ or that $f(x + h)$ represented the output of $x + h$).

6.3 Functions

The data analysis table for functions is made similarly to the ones for composition and $f(x + h)$. Student thinking about function can be revealed in numerous discussions that do
not necessarily revolve around functions, such as, function composition, $f(x + h)$ evaluation, derivatives, or limits, etc. For the purposes of this study, analysis of students’ conceptions about function is focused on student explanations of what is a function, and their discussions on composition and $f(x + h)$ evaluation.

Students during the function conversations spoke generally about functions and discussed conditions under which situations were considered functions. Therefore, the action row of Table XXXIV shows only the third bullet of the left column of Table II in Chapter 2 (p. 15). The first two bullets of the left column of Table II were relocated to a new transition level between action and process in Table XXXIV; although, students spoke in generality about functions, they were still restricted to equations. This generality forced them out of the action stage, but the restriction to equations kept the students from being categorized as having process conceptions. As discussed in Chapter 2, the last two bullets of Table XXXIV reflect the additional definitions that were added to the definitions for this conception.
**TABLE XXXIV: APOS THEORY FOR FUNCTION**

<table>
<thead>
<tr>
<th>Stage</th>
<th>General Definition</th>
<th>Definition Applied to Function</th>
<th>Example from Student</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Action</strong></td>
<td>“set of step-by-step instructions performed explicitly to transform physical or mental objects” (Dubinsky et al., in press)</td>
<td>explicitly making a calculation for solving one variable for another (Dubinsky and Harel, 1992)</td>
<td>Lian explicitly solves for $y$ when asked if $y^2 = x$ is a function.</td>
</tr>
<tr>
<td><strong>Action/Process</strong></td>
<td>generality in the definition of function, but explicitly connects to equations/expressions</td>
<td>• explicit mention of input/output/transformation missing (Breidenbach et al., 1992)</td>
<td>• “When you can put a value into it and you can get another value out”                                   -Drew</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• if above were present, but procedure tied to expression or equation (Breidenbach et al., 1992)</td>
<td>• “It’s almost like a machine you can put something into it and get an output, so you’ll have an equation that you can describe something and then depending on what you put into it, you get something else out of it.” -Pierre</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• attention to domain, i.e., has to be defined for all inputs</td>
<td>• “[It would be a function] depending on the input . . .”                                               -Sterne</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• recognize function maps input to unique output; restricted to equation</td>
<td>• “for one variable, you would have one solution, not two.”                                            -Ray</td>
</tr>
<tr>
<td><strong>Process</strong></td>
<td>• procedure is entirely imagined</td>
<td>• imagine function (procedurally same as action)</td>
<td>Not Found</td>
</tr>
<tr>
<td>Stage</td>
<td>General Definition</td>
<td>Definition Applied to Function</td>
<td>Example from Student</td>
</tr>
<tr>
<td>---------</td>
<td>---------------------------------------------------------------------------------</td>
<td>----------------------------------------------------------------------------------------------------------------------------</td>
<td>----------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Process</td>
<td>(Cont.)</td>
<td>• “talk about [solving one variable for another] without actually obtaining the expression is probably displaying at least the beginning of a process conception of function” (Dubinsky and Harel, 1992, p. 93)</td>
<td>• “You can make ( y ) in terms of ( x ).” – Jo</td>
</tr>
<tr>
<td></td>
<td>• external stimuli not required</td>
<td>• recognition that functions have to map inputs to unique outputs (not restricted to equations)</td>
<td>• “a function would only have ... one output ... for every input” – Sterne</td>
</tr>
<tr>
<td></td>
<td>+ can be combined with other processes (Asiala et al., 1996; Breidenbach et al., 1992; Dubinsky et al., in press; Dubinsky and Mcdonald, 2002)</td>
<td>• does not need cue</td>
<td>• NA</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+ has at least process view of (pointwise or uniform) function composition (Asiala et al., 1996; Breidenbach et al., 1992)</td>
<td>+ See Function Composition Data Table XXIX for examples</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+ not restricted to expression/ equation, e.g., graphs, ordered pairs</td>
<td>+ “points... (4,5), (6,7) [and (8,1)]” – Nat</td>
</tr>
<tr>
<td>Totality</td>
<td>recognition of concept as an entity, but does not act on it</td>
<td>recognition that the function is an entity, e.g., a set of output values</td>
<td>Not found</td>
</tr>
<tr>
<td>Object</td>
<td>• realization that process can be acted on with previously established actions or processes (Asiala et al., 1996; Dubinsky and Mcdonald, 2002)</td>
<td>• discusses actions or processes on functions in generality</td>
<td>• “add, subtract, multiply, divide, um, limits, derivatives. Um, graph- ing.” – Jo</td>
</tr>
<tr>
<td></td>
<td>• explicitly perform a previously established action or process on the process (Asiala et al., 1996; Breidenbach et al., 1992)</td>
<td>• has process view of uniform function composition</td>
<td>• See Function Composition Data Table XXIX for examples</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• views ( f(x + h) ) as a horizontal transformation</td>
<td>• “shifts it. Either, I think to the right, or something.” – Lauri</td>
</tr>
</tbody>
</table>
The process conception of function has the usual two requirements and one indication, but in the first requirement, there are three levels. The first two were found in the literature, and the inclusion of the third comes directly from my study. Indications of a process understanding of function come from observing students’ views of function composition. The second indication is an extension of the statement that ordered pairs is a bellwether for a process conception (Dubinsky and Harel, 1992). Other representations, such as graphs, are included as well.

In the totality stage, there was no concrete evidence of the participants’ thinking about functions as an entity. There were students who referred to them as “something”, or “it”, but none that referred to them as sets. In the last stage, there are two requirements of an object conception: realization that processes can be acted on, and explicitly acting on them. The second requirement has two levels that come directly from this study. Since compositions and \( f(x + h) \) are transformations of functions as objects, how students viewed these concepts gives insight into the strength of an object conception of function.

Table XXXV shows the assessment tool created from Table XXXIV. Students in this part of the study all reached the process stage of understanding functions. The ability to compose two functions uniformly moved eight of the ten students into the object conception of function. Details can be found in Appendix F.

6.4 Closing Remarks

This chapter can provide researchers a sense of what constitutes an APOS conception of function, composition, and \( f(x + h) \). For other mathematical concepts, it also serves as a model to create data analysis tables to classify students into APOS conceptions stages. Additionally,
TABLE XXXV: ASSESSMENT OF ACTION, PROCESS, TOTALITY, OBJECT CONCEPTIONS OF FUNCTION

<table>
<thead>
<tr>
<th>Student</th>
<th>Action</th>
<th>A/P</th>
<th>Process</th>
<th>Totality</th>
<th>Object</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>1</td>
<td>2</td>
<td>1++</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>0</td>
<td>1</td>
<td>1++</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Ray (A)</td>
<td>0</td>
<td>1</td>
<td>1++</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>0</td>
<td>2</td>
<td>1++</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Drew (D)</td>
<td>0</td>
<td>1</td>
<td>2++</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Turner (C)</td>
<td>0</td>
<td>1</td>
<td>2++</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Jo (A)</td>
<td>0</td>
<td>1</td>
<td>2++</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Nat (-)</td>
<td>0</td>
<td>1</td>
<td>1++</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Sterne (F)</td>
<td>0</td>
<td>2</td>
<td>2++</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>0</td>
<td>2</td>
<td>1++</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

strength levels were added to some stages since students in this study frequently showed evidence of fulfilling one or more criteria of a stage, but not necessarily all of the criteria.

This chapter only offers a glimpse into creating a systematic tool for assessing students' mathematical conceptions using APOS, rather than providing a complete solution to the complexities in understanding students' thinking. For example, the values assigned to students in each stage need to be further developed to accurately measure the strengths of students' conceptions. However, the sample size of the group of students used to develop these data analysis tables was too small to achieve this. Much more research must be done to move towards achieving comprehensive data analysis tools.
CHAPTER 7

IMPLICATIONS

This study exposed and categorized the errors students made when evaluating the DQs and also exposed student thinking of functions, composition, and DQ evaluations as revealed through discussion of these errors. What follows in this chapter are a four implications teaching that came out of our understanding of student conceptions of functions, composition, and difference quotients.

Function Composition to Aid in Understanding $f(x + h)$

Findings on the effectiveness of using function composition to evaluate $f(x + h)$ shows the concept can be a valuable learning tool to help students transition out of an action conception into a process conception, or if they already possess a process conception of function, it can help strengthen the process conception; that is, they no longer view evaluation of $f(x + h)$ as an action of replacing or substituting $x$ with $x + h$.

During Interview 2, all students were asked this question:

Can you use function composition to evaluate $f(x + h)$? In other words, can you define a $g(x)$ such that $f(g(x)) = f(x + h)$?

(a) With your $g(x)$ from above, can you give $f(g(x))$ for $f(x) = \tan x + \frac{1}{x} - x^2$?
(b) How does this compare with what you did before?
Nat took the longest to complete this construction. After she succeeded, I asked her if she saw a connection between this question and the \( f(x + h) \) evaluation. She said, “[B]asically you were saying that \( x + h \) \( [f(x + h)] \) is the composition, it’s not what I said originally. [Before] I just said you just replaced \([x \text{ with } x + h]...Because I didn’t know—or [thought] you just substitute in. It’s actually a composition. I get it,” (Interview 2, #00:05:24.6#).

It is clear here that she has transitioned out of thinking about \( f(x + h) \) as an action, or out of an early process conception. This demonstrates that giving students a construction task that connects \( f(x + h) \) evaluation with function composition can be useful to help them transition into stronger conception categories.

Awareness of Intended Meaning v. Interpreted Meaning (Spoken)

“My Words
Come from my life’s experiences
Your understanding
Comes from yours

Because of this
What I say
And what you hear
May not be the same”
-Javan

It was shown in the previous sections that Pierre and Lian recalled their instructors telling them to “plug it in” when they were evaluating the DQs. Turner, who did not make any of the errors analyzed in Phase 2 on her quizzes nor during her interviews, also mentioned instructor’s

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\(^1\)The times ranged from 6 seconds to 9.5 minutes. See Table XLVI in Appendix F.
spoken language. When I showed her another student’s work (Sample 3; Appendix D.3) which had errors $f(x + h)^2$ and $f\left(\frac{1}{x + h}\right)$ for the $f(x + h)$ in the DQ$_h$, Turner recalled her TA’s statement to the class in response to a student’s question. Below she tells what was said:

Turner: [H]e was like, ‘just plug it in’. So I guess that’s just what they were doing. They were literally just plugging it in. So just plugging in this whole um, equation into this function. #00:26:10.9#

; Turner: [L]ike, I guess the $f(x + h)$, they saw it as, it was supposed to be squared. #00:26:34.8#

In the “plug it in” language, it is not clear what students are supposed to plug in, and where this plugging in should happen.

It is beyond the scope of this study to confirm whether the instructors told students to “plug it in” with no details of what to plug in and where to plug it. Rather, this study seeks to bring awareness to the fact that whatever the intended instructions, they were interpreted by students as “plug it in”. As we have seen in this study, these interpretations affected how the DQ’s were evaluated.

This study emphasizes the need to be deliberate about the words we choose when teaching concepts to students and to be aware of their possible interpretations. For example, if we, as educators, are specific about the objects that are being acted on, and which processes act on these objects, students will gain clarity as well.

Awareness of Intended Meaning v. Interpreted Meaning (Written)

The previous discussion addressed the impact of the interpretation of instructors’ spoken language on student thinking of DQ evaluation; this section reports on the impact of interpre-
tation of instructors’ written language. It would be wrong to assume that I place blame on the instructor, rather I attempt to bring awareness to the impact of the written meanings we convey to students on their understanding.

Recall that Ray made a comment regarding the lack of understanding why his instructor wrote \( \lim_{x \to 0} \) “in front of each step” and how this lack of understanding reinforced him to write “\( f() \)” in front of the given function to make the error \( f(x^2) \). Rather than viewing \( f(x^2) \) as addition of the \( f \) to \( x^2 \) as in Ray’s statement, Lian made a comment about how she did not understand why her instructor “gets rid of the \( f \)” to get \( x^2 \). These two comments demonstrate an underdeveloped understanding of the function notation. To strengthen this understanding, I offer the following promising teaching strategy.

Allow Students to Explore Notation

The word function first appeared in 1694, courtesy of Leibniz, but it was not until 40 years later in 1734 that Euler would introduce the \( f(x) \) notation (Eve and Newsom, 1958). Despite this duration and the fact that the notation was created to suit Euler’s purposes, students are expected to take ownership of this notation and to understand its meaning shortly after its introduction. Especially for a traditional course on calculus, understanding the notation is no doubt important to the success of students in the class; it is through the use of notation that students’ understandings are ultimately assessed. As such, it is important to help students understand the notation in which they are expected to communicate their mathematical ideas. Research from this study shows the value in allowing students to explore the notation.
A conversation with Lauri showed that she repeatedly changed $f(x)$ to $y$. This occurred while discussing why some students made errors with the DQ$_a$. As a solution, she offered, “say that this was $y = \frac{1}{x}$, I think it would be easier to differentiate between the two,” (Interview 2, #00:15:44.5#). To demonstrate the limitations of always changing $f(x)$ for $y$, I asked her how this new view would help with DQ$_a$ evaluation. She explained that the numerator, “would [be], $y$ minus—I don’t know how you would do that [circles the $f(a)$ required in the DQ$_a$] but—[writes $y - f(a)$],” (Interview 2, #00:16:36.3#). Notice that after taking away the $f$ notation, she re-introduced it. She tried to reconcile the mixed notations here and said, “say that this was just $f$ [of] $y$ [puts “$f(\cdot)$” in front of $y$ so that it’s $f(y) = \frac{1}{x}$]. Um. I don’t know,” but she quickly abandoned the idea and returned to setting $y$ equal to $\frac{1}{x}$. She then repeated the actions of what she just did for the numerator of DQ$_a$ and stated that she did not know how to handle $f(a)$ if she called $\frac{1}{x}, y$ instead of $f(x)$.

After probing Lauri on what she did not like about the notation, she said “the actual function itself usually deals with $x$ as the variable, so say that the $x$ that was in the $f(x)$ part, was notated as something else. I think it would have been easier,” (Interview 2, #00:17:51.9#). It is clear here that she did not understand the dependence of $f(x)$ on the $x$. However, she did understand the dependence of $y$ on $x$: “if you think about it as $y$, and this is $x$ then, in order to get $y$ you would need to plug in $x$” (Interview 2, #00:18:46.3#). She has demonstrated that she thinks of $f(x)$ as equal to $y$ and that $y$ is equal to $\frac{1}{x}$, but she does not make the connection that $f(x) = \frac{1}{x}$.
Since she stated that “it would have been easier” to change the $x$ in the $f(x)$ notation to something else, I asked her to do so. She said, “say this [the independent variable] was just any other variable that isn’t common. I don’t know, like $p$ [writes $f(p) = \frac{1}{x}$].” To demonstrate the problem with this notation, I asked her to tell me what $f(2)$ would be. First, she said to plug in 2, and when I pointed out that $2 = p$, she said, “That’s why I’m saying I don’t think it would work. That’s probably why these [x’s in the notation $f(x)$ and on the right hand side of the equation] are the same,” (Interview 2, #00:19:36.5#).

In the beginning of this episode, she modified the function notation to exclude the $f$ by replacing $f(x)$ with $y$. By allowing her to explore the limitations of this change, she noticed that the substitution could not be adequately used to represent $f(a)$. As a result, she reintroduced the $f$, but suggests that the $x$ in the $f(x)$ should be renamed as $p$. By permitting her to do this, she began to construct the idea of why the $x$ in the $f(x)$ has to be the same as the $x$’s in the expression. This research moment demonstrates the promise of allowing students to explore the notation in order to strengthen their understandings of the function notation. Therefore, it should be explored furthered as its effectiveness as a teaching strategy.
CHAPTER 8

CONCLUSION

Phase 1 of this study surfaced the areas of struggle for students during DQ evaluations and Phase 2 exposed student thinking about function, composition and DQ evaluations as revealed by these areas of struggle. Since the details of the conclusions can be found in Section 4.3 and Section 5.4, respectively, only a summary will be provided here.

Student thinking on these mathematical concepts put forth insight into why students struggled with the DQ evaluations. In particular, it was shown that students viewed the DQ’s as formulas that accept functions as inputs. Difficulty in coordinating inputs represented as $x$, and also as $f(x)$ caused student errors. However, students did not have trouble coordinating both $x$’s and functions as inputs in routine function composition where functions are inputs. This dissonance is also reflected in the lack of correlations between the scores of DQ evaluation problems and function composition problems, suggesting that students do not use knowledge about functions across problems. Conversations with students showed similar results.

The qualitative phase showed that students’ thinking of function composition, and $f(x + h)$ were disjoint. Students seem to have medium to strong notions of function, composition, and $f(x + h)$ as isolated topics, rather than a strong notion of the connections between the concepts. However, after asking the question that bridges composition and $f(x + h)$, students began describing student errors in terms of composition, making a connection between these two previously isolated concepts.
Another example where students did not always make full use of the knowledge they possess comes from the $f(a)$ evaluation. Students who did not evaluate $f(a)$ showed that they treated $f(x)$ and $f(a)$ as two different entities. That is, students are not using their concept definitions of $f(x)$ to understand $f(a)$.

Producing data analysis tools was not part of the original research, but came out of my difficulty in using APOS to assess participants’ conceptions of function, function composition and evaluation of $f(x+h)$. The data analysis tools that came out of the current research is a first attempt at systematizing the use of the theory. Further research is needed to reduce the gap between the intended theory and the enacted theory.

Findings from this study on errors can be used to inform teaching practices, curriculum development, or to further research. Educators can help students make connections between DQ evaluation and function composition, but most importantly become aware that our language does not always get interpreted as intended.

As a closing observation, it is common in the literature to find statements in the nature of: even the most high-achieving students display misconceptions. On the contrary, what is less commonly stated is that students who fail classes show advanced understanding of some of the topics of the class. This was true in this study; although the nature of this study focused on the errors students made, and thus focused on their weaknesses, additional data that did not contribute to the purpose of this study showed evidence of advanced understanding of particular mathematical ideas. This suggests that there are factors beyond intellectual abilities that affect student achievement in calculus. In fact, in preliminary national study of college
calculus found the most common reason students are switching out of STEM majors was due to “bad experiences,” as opposed to “bad grades” or “didn’t understand concepts” (Rasmuseen, 2012). This is encouraging because we, as educators, have control over not only changing, but improving our students’ learning experiences.
APPENDICES
Appendix A

PHASE 1 QUIZZES

A.1 Quiz 1

1. Set up the difference quotient,

\[
\frac{f(x) - f(a)}{x - a},
\]

for each of the following functions:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin x \)

2. Please evaluate \( f(2 + h) \) for each of the same functions above:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin x \)
A.2 Quiz 2

1. Set up the difference quotient,

\[
\frac{f(x + h) - f(x)}{h},
\]

for each of the following functions:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin x \)
A.3 Quiz 3

1. Compose \( f \) and \( g \), i.e. give \( f(g(x)) \), for:

(a) \( f(x) = x^3 \) and \( g(x) = x + 1 \)

(b) \( f(x) = \frac{2}{x} \) and \( g(x) = 3 + x \)

(c) \( f(x) = \tan x \) and \( g(x) = x - 1 \)
Appendix B

PHASE 2: INTERVIEW GUIDES

B.1 Interview Guide

Background Questions - (20 Minutes)

1. How old are you?
2. What year of study are you in?
3. Why are you taking this class?
4. What do you consider to be your gender?
5. What do you consider your ethnicity and race? (Here I will ask about family too, i.e. occupations of parents/guardians, highest degrees obtained).
6. What is your major?
7. Are you taking this class to fulfill requirements for your major or is it an elective?
8. What other math classes have you taken?
9. What do you think about Calculus?

B.1.1 Definition Questions on Functions (Research Question 2) (mins)

1. What is a function? (2 mins)
2. Give examples of functions. (5 mins)
   (a) Why is this a function?
   (b) How does this match your definition of a function?
3. Describe the different ways a function can be represented. (For each different representation give an example to the student and ask it’s a function or not.)
4. What is a domain? (4 mins)
   (a) What are some possible domains of functions?

B.1.2 Function Notation Questions - (minutes)

1. Write \( f(x) \) on a piece of paper for them and ask “We use this notation a lot, can you explain what this notation means?” Make sure they address all these points:
   (a) What does the \( x \) represent? (3 mins)
   (b) What do the parentheses represent? (2 mins)
   (c) Where else in math have you seen parenthesis? (1 min)
Appendix B (Continued)

(d) Does it mean the same thing in $f(x)$ notation? (1 min)
(e) What does the $f$ represent? (2 mins)
(f) What operations can we do with $f(x)$? (2 mins)
(g) Now how does what you are saying about $f(x)$ apply to $f(x + h)$ (4 mins)?

2. For the given graph, label an arbitrary $x$-value on the graph and call it $a$. Mark also its image (the value of $f(a)$). Draw and label the point $(a, f(a))$. Let $b = a + 2$. Draw and label the point $(b, f(b))$. (5 mins)
Appendix B (Continued)

B.2 Interview II Guide

B.2.1 \( f(x + h) \) Questions - (10 minutes)

1. A classmate asks you how to find \( f(x + h) \) for \( f(x) = x^2 + x - 5 \). How do you explain it to them?

2. Ok, so, now how do I find \( f(x + h) \) for \( f(x) = \cos(x) \)?

B.2.2 Function Composition Questions

1. What is function composition?

   (a) When have you seen function composition? (In what context?)

   (b) What is the domain of this example of function composition that you gave?

2. Can you use function composition to evaluate \( f(x + h) \)? In other words, can we define a \( g(x) \) such that \( f(g(x)) = f(x + h) \)?

3. With your \( g(x) \) from above, can you give \( f(g(x)) \) for \( f(x) = \tan(x + 1/x - x^2) \)?

B.2.3 Student Error Questions

1. Take a look at sample 1 number 1 (Appendix D.1 and D.2). What do you think about this solution? Is it correct? Is it incorrect? How did the student come up with this answer? What do you think about this denominator?

2. Take a look at sample 3 number 3 a), b) and c) (Appendix D.3). What do you think about this solution? Is it correct? Is it incorrect? How did the student come up with this answer?

3. Take a look at sample 4 number 2 (Appendix D.4). What do you think about this solution? Is it correct? Is it incorrect? How did the student come up with this answer?
C.1 Questions for the First Interview

In the first interview, I will begin by asking background questions. The interview guide for this can be found in Appendix refapp:int1. I will also be asking questions on definitions that are outlined below. These questions are asked with the understanding that they are surface level questions. These surface questions are meant to expose the language the students use when talking about functions with the purpose of facilitating our conversation. Though they may reveal deep misconceptions, it is not expected as a definite result. The questions during the second interview are designed to elicit deeper understanding by asking students to perform specific tasks.

C.1.1 Definition Questions on Functions (Research Question 2) (16 mins)

Hypothesis: Students are having trouble with evaluating $f(x + h)$ because they have a weak notion of function. In other words, students have trouble with one or more of the following:

1. understanding function as process and object.
2. what a function is.
3. function notation.

Need to know:

1. What is their conceptual and procedural understanding of function?

Questions and their purposes

1. What is a function? (2 mins)

Students need to understand that a function can be representative of both a process and an object, and also be able to transition between the two conceptions for different situations. Much like Breidenbach et al. (1992), I want to find out the students’ understanding of a function so that I can analyze whether they view it as process and/or object. Understanding which conception students have is especially important when composing functions because functions as processes act on functions as objects. Some examples of responses to this question from Briedenbach et al. (1992):

(a) A function is something that evaluates an expression in terms of $x$.
(b) A function is an equation in which a variable is manipulated so that an answer is calculated using numbers in place of that variable.
Appendix C (Continued)

(c) A function is a statement that when given values will operate with these values and return some result.
(d) A function is some sort of input being processed, a way to give some sort of output. (p. 252)

2. Give examples of functions. (5 mins)
   (a) Why is this a function?
   (b) How does this match your definition of a function?
      To give me a deeper understanding of their definition of a function.

3. Describe the different ways a function can be represented.
   To draw out different representations of functions

4. What is a domain? (4 mins)
   (a) What are some possible domains of functions?
      This will help me gauge their understanding of domain.

C.1.2 Function Notation Questions - (18 minutes)

Hypothesis: Students do not understand function notation, particularly what each piece of the notation represents and therefore have trouble with evaluating $f(x + h)$.

Need to know:

1. What is their understanding of function notation?
2. What is their conceptual understanding of $f(x + h)$ compared to $f(x)$?
3. What is their conceptual understanding of $x + h$?

Questions and their purposes

1. Write $f(x)$ on a piece of paper for them and ask "We use this notation a lot, can you explain what this notation means?" Make sure they address all these points:
   (a) What does the $x$ represent? (3 mins)
      Find out what the students think $x$ is? Do they talk about domain?
   (b) What do the parentheses represent? (2 mins)
      Do they think this is multiplication?
Appendix C (Continued)

(c) What does the $f$ represent? (2 mins)
Are they taking the $f$ that was given and multiplying it with the argument ($x$)?

(d) What operations can we do with $f(x)$? (2 mins)
Want to know which actions they are familiar with, e.g. addition, taking limit, composition.

(e) Now how does what you are saying about $f(x)$ apply to $f(x+h)$ (4 mins)?
I really want to know if students have a second layer conception of $x+h$. Can students abstract the idea of the varying nature of $x$, or do they have to plug in numbers for $x$ in order to understand it? And what does the $h$ mean to them?

2. For the given graph, label an arbitrary $x$-value on the graph and call it $a$. Mark also its image (the value of $f(a)$). Draw and label the point $(a, f(a))$. Let $b = a + 2$. Draw and label the point $(b, f(b))$. (5 mins)
I want to find out what $f(x+h)$ means to them. In particular, I want to see how students understand the relationship of $x$ and $x+h$ for any given $x$. Then I want to see how students understand $f(x)$ and its relationship to $f(x+h)$ for the given $x$.
If they talk about this problem with numbers, then they have a first layer conception, if they talk about it with the constants given, then they have reached the second layer.

C.2 Questions for the Second Interview

C.2.1 $f(x+h)$ Questions - (10 minutes)

Hypothesis: Students are having trouble with evaluating $f(x+h)$ because they have an erroneous procedural method of evaluating $f(x+h)$. Also, students do not understand the notation of $f(x+h)$.

Need to know:

1. What is their method?

Questions and their purposes

1. A classmate asks you how to find $f(x+h)$ for $f(x) = x^2 + x - 5$. How do you explain it to them?
   To understand their procedural method.

2. Ok, so, now how do I find $f(x+h)$ for $f(x) = \cos(x)$?
   To see if there is a difference in procedure across different types of functions.
C.2.2 Function Composition Questions

*Hypothesis:* Students do not understand that \( f(x + h) \) can be thought of as a composition of two functions.

*Need to know:*

1. What is their conceptual and procedural understanding of function composition?
2. What happens when it is pointed out to them that \( f(x + h) \) can be broken down into a composition of two functions?

Questions and their purposes

1. What is function composition?
   *To draw out their conceptual knowledge of function composition. (The procedural part has been asked on the quiz).*

2. Can you use function composition to evaluate \( f(x + h) \)? In other words, can we define a \( g(x) \) such that \( f(g(x)) = f(x + h) \)?
   *I want to know if students can understand \( f(x + h) \) as a function composition question.

3. With your \( g(x) \) from above, can you give \( f(g(x)) \) for \( f(x) = \tan x + 1/x - x^2 \)?
   *Compare the way the students answer this problem with questions 1 and 2 of Section C.2.1. I want to know if they have trouble with one and not the other? When told that it’s function composition, I want to know if they can evaluate \( f(x + h) \) easily.*

C.2.3 Student Error Questions

This section is based on actual student errors. If students themselves made the following errors, I will show them their own paper and ask them how they came up with that answer. If the student did not make that type of error, I will show them another student’s work and ask them how they think the student came up with the answer.

1. A student was asked to set up a difference quotient,

\[
\frac{f(x) - f(a)}{x - a},
\]

for the function \( f(x) = x^2 \)

They gave a solution of \( \frac{f(x)^2 - f(a)}{x - a} \).

What do you think about this solution? Is it correct? Is it incorrect? How did the student come up with this answer?
Appendix C (Continued)

2. A student was asked to set up a difference quotient,

\[ \frac{f(x) - f(a)}{x - a}, \]

for the function \( f(x) = \frac{1}{x} \).

(a) He/She gave a solution of \( \frac{f\left(\frac{1}{x}\right) - f(a)}{x - a} \).

What do you think about this solution? Is it correct? Is it incorrect? How did the student come up with this answer?

(b) Another student gives a solution of \( \frac{\frac{1}{x} - f(a)}{x - a} \).

What do you think about this solution? Is it correct? Is it incorrect? How did the student come up with this answer?

3. A student was asked to set up,

\[ \frac{f(x + h) - f(x)}{h}, \]

for the following function \( f(x) = \frac{1}{x} \)

He/She gave a solution of \( \frac{f\left(\frac{1}{x} + h\right) - f(a)}{x - a} \).

What do you think about this solution? Is it correct? Is it incorrect? How did the student come up with this answer?
Appendix D

SAMPLES SHOWN TO INTERVIEWEES

D.1 Sample 1 (Sample 2 for Nat)

This is the new sample 1 chosen to include everything the old sample 1 (below) had, but also includes the denominator error.

1. Set up the difference quotient,

\[
\frac{f(x) - f(a)}{x - a},
\]

for each of the following functions:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin x \)

D.2 Sample 1 for Nat and Ray

Nat and Ray got this version originally, but I realized I wanted the participants to talk about the denominator errors, so I chose a new Sample 1 for the rest of the group to talk about.

2. Setup the difference quotient:

\[
\frac{f(x) - f(a)}{x - a}
\]

for each of the following functions:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin(x) \)
D.3 Sample 3

3. Set up the difference quotient,

\[
\frac{f(x+h) - f(x)}{h},
\]

for each of the following functions:

(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

D.4 Sample 4

3. Please evaluate \( f(2+h) \) for each of the same functions above:

(a) \( f(x) = x^2 \)
\[
\frac{f(2+h)^2 - f(2)}{x-2}
\]

(b) \( f(x) = \frac{1}{x} \)
\[
\frac{f\left(\frac{2+h}{2}\right) - f(2)}{x-2}
\]

(c) \( f(x) = \sin x \)
\[
\frac{f\left(\sin(2+h)\right) - f(2)}{x-2}
\]
Appendix E
STUDENT QUIZZES

E.1 Lian: Quiz 1

Episode III. Student Error Questions

2. Setup the difference quotient:
\[
\frac{f(x+h) - f(x)}{h} = \frac{f(x) - f(a)}{x - a}
\]
for each of the following functions:
(a) \( f(x) = x^2 \)
\[
\frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h
\]
(b) \( f(x) = \frac{1}{x} \)
\[
\frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = -\frac{1}{x(x+h)}
\]
(c) \( f(x) = \sin(x) \)
\[
\frac{\sin(x+h) - \sin(x)}{h}
\]

3. Please evaluate \( f(2 + h) \) for each of the same functions above:
(a) \( f(x) = x^3 \)
\[
2x^3 + 3x^2 h + 3xh^2 + h^3
\]
(b) \( f(x) = \frac{1}{x} \)
\[
\frac{1}{(x+h)^2}
\]
(c) \( f(x) = \sin(x) \)

4. (a) For the first function above, \( f(x) = x^2 \), use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \).
\[
2x + h = 2(3) + h = 6 + h
\]

(b) Find the tangent line to \( f(x) \) at \( x = 3 \) using the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(1) \).
\[
\frac{1}{(x+h)^2} = \frac{1}{(3+h)^2}
\]

(c) For the second function above, \( f(x) = \frac{1}{x} \) use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \).
E.2 Lian: Quiz 2

III. Compute the rate of change of the surface area $A$ of a cube with respect to volume $V$. Be careful, this is similar to problem 6 in the textbook, but not the exact same problem. Recall the volume $V$ of a cube is given by $V = s^3$ where $s$ is side length, while surface area $A$ is given by $A = 6s^2$.

\[
\begin{align*}
A &= 6s^2 \\
\sqrt{\frac{A}{6}} &= \sqrt[3]{s^2} \\
S &= \sqrt{\frac{A}{3}} \\
V &= (\frac{A}{6^2} \cdot \frac{1}{6})^3 = \frac{1}{2} A - \frac{s^3}{6^2} \cdot \frac{V}{3}
\end{align*}
\]

Episode III. Student Error Questions

IV. Set up the difference quotient:

\[
\frac{f(x + h) - f(x)}{h}
\]

for each of the following functions:

1. $f(x) = x^2$

\[
\frac{(x + h)^2 - x^2}{h} = \frac{x^2 + 2hx + h^2 - x^2}{h} = \frac{2hx + h^2}{h} = \frac{h(x + h)}{h} = x + h
\]

2. $f(x) = \frac{1}{x}$

\[
\frac{\frac{1}{x + h} - \frac{1}{x}}{h} = \frac{\frac{x}{(x + h)(x)} - \frac{x + h}{(x + h)(x)}}{h} = \frac{1}{x^2 + hx}
\]

3. $f(x) = \sin x$

\[
\frac{(\sin(x + h)) - \sin x}{h}
\]
Appendix E (Continued)

E.3 Pierre: Quiz 1

2. Setup the difference quotient:

\[
\frac{f(x) - f(a)}{x - a}
\]

for each of the following functions:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin(x) \)

3. Please evaluate \( f(2 + h) \) for each of the same functions above:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin(x) \)

4. (a) For the first function above, \( f(x) = x^2 \), use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \).

\[
\lim_{h \to 0} \frac{f(3+h)^2 - 3^2}{h} = \frac{(3+h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h} = \frac{h(6 + h)}{h} = 6 + h
\]

(b) Find the tangent line to \( f(x) \) at \( x = 3 \) use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f''(1) \).

\[
y - 9 = 6(x - 3)
\]

(c) For the second function above, \( f(x) = \frac{1}{x} \) use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \).

\[
f'(3) = \left( \frac{1}{3} \right) - 3
\]
E.4 Pierre: Quiz 2

III. Compute the rate of change of the surface area \( A \) of a cube with respect to volume \( V \). Be careful, this is similar to problem 6 in the textbook, but not the exact same problem. Recall the volume \( V \) of a cube is given by \( V = s^3 \) where \( s \) is side length, while surface area \( A \) is given by \( A = 6s^2 \).

\[
A = 6s^2 \quad \frac{A}{6} = s^2 \quad s = \sqrt[3]{\frac{A}{6}} \quad \frac{\sqrt[3]{\frac{A}{6} + \sqrt[3]{\frac{1}{6}}}^3}{6} = \frac{1}{6} \left( \frac{1}{A^2} \right)^{\frac{3}{2}}
\]

IV. Set up the difference quotient:

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

for each of the following functions:

1. \( f(x) = x^2 \)

\[
\frac{x^2 + 2hx + h^2 - x^2}{h} = \frac{2hx + h^2}{h} \quad \frac{2xh + h^2}{h} = \frac{h(2x+h)}{h} = 2x + h
\]

2. \( f(x) = \frac{1}{x} \)

\[
\frac{1}{x(h)} - \frac{1}{x} = \frac{1}{x} - \frac{1}{x}
\]

3. \( f(x) = \sin x \)

\[
\frac{\sin(x+h) - \sin x}{h}
\]
Appendix E (Continued)

E.5 Ray: Quiz 1

2. Setup the difference quotient:
\[ \frac{f(x) - f(a)}{x-a} \]
for each of the following functions:
(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin(x) \)

3. Please evaluate \( (2+h)^7 \) for each of the same functions above:
(a) \( f(x) = x^2 \)
\( (2+h)^2 = \frac{(2+h)(2+h)}{4+4h+h^2} \)
(b) \( f(x) = \frac{1}{x} \)
\( \frac{1}{2+h} \)
(c) \( f(x) = \sin x \)
\( \sin(2+h) \)

4. (a) For the first function above, \( f(x) = x^2 \), use the limit definition of the derivative 
(not the power rule, though you may use it to check your work) to find \( f'(3) \).
\[ \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{6h^2 + 6h}{h} = 6h \]
\( f'(x) = 2x \)
\( f'(3) = 6 \)

(b) Find the tangent line to \( f(x) \) at \( x = 3 \). Use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(1) \).
\[ \frac{(1+h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h \]
\( y - 1 = 2(x-3) \)
\( 1 = 2x - 6 \)
\( 2x = 7 \)
\( x = \frac{7}{2} \)

(c) For the second function above, \( f(x) = \frac{1}{x} \), use the limit definition of the derivative 
(not the power rule, though you may use it to check your work) to find \( f'(3) \).
\[ \frac{\frac{1}{(3+h)} - \frac{1}{3}}{h} = \frac{3 - 3 - h}{h(3+h)(3)} = \frac{-h}{h(3+h)(3)} = \frac{-1}{(3+h)(3)} \]
\( f'(3) = -\frac{1}{9} \)
Appendix E (Continued)

E.6 Ray: Quiz 2

Dear student,

This problem was received one day before the due date. Not sure of the answer.

III. Compute the rate of change of the surface area $A$ of a cube with respect to volume $V$. Be careful, this is similar to problem 6 in the textbook, but not the exact same problem. Recall the volume $V$ of a cube is given by $V = s^3$ where $s$ is side length, while surface area $A$ is given by $A = 6s^2$.

$$L = s^2$$
$$\frac{A}{2} = s \frac{2}{3}$$
$$\frac{V}{3} = s (\frac{1}{2} \sqrt[2]{A})$$

IV. Set up the difference quotient:

$$\frac{f(x+h) - f(x)}{h}$$

for each of the following functions:

1. $f(x) = x^2$

$$\frac{f(x+h)^2 - f(x)^2}{h} = \frac{(x+h)^2 - x^2}{h}$$

2. $f(x) = \frac{1}{x}$

$$\frac{\frac{1}{(x+h)^2} - \frac{1}{x}}{h}$$

3. $f(x) = \sin x$

$$\frac{\sin(x+h) - \sin x}{h}$$
Appendix E (Continued)

E.7 Lauri: Quiz 1

2. Setup the difference quotient:

\[
\frac{f(x) - f(a)}{x - a}
\]

for each of the following functions:

(a) \( f(x) = x^2 \):

\[
\frac{x^2 - a^2}{x - a} = \frac{(a+h)^2 - a^2}{h} = h^2 + 2ah + a^2
\]

(b) \( f(x) = \frac{1}{x} \):

\[
\frac{1}{x} - \frac{1}{a} = \frac{1}{x-a}
\]

(c) \( f(x) = \sin(x) \):

\[
\frac{\sin(x) - \sin(a)}{x - a}
\]

3. Please evaluate \( f(2 + h) \) for each of the same functions above:

(a) \( f(x) = x^2 \):

\[
(\alpha + h)^2 = h^2 + 4h + 3
\]

(b) \( f(x) = \frac{1}{x} \):

\[
\frac{1}{2 + h} = \frac{1}{2h}
\]

(c) \( f(x) = \sin x \):

\[
\sin(2 + h) = \sin(3 + h)
\]

4. (a) For the first function above, \( f(x) = x^2 \), use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \).

\[
f'(3) = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \frac{h^2 + 6h + 9 - 9}{h} = 6 + \frac{6h}{h} = 6 + 6 = 12
\]

(b) Find the tangent line to \( f(x) \) at \( x = 3 \); use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \).

\[
y - 9 = 6(x - 3)
\]

(c) For the second function above, \( f(x) = \frac{1}{x} \), use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \).

\[
f'(3) = \lim_{h \to 0} \frac{\frac{1}{3 + h} - \frac{1}{3}}{h} = \frac{3}{9 + 3h} + \frac{-3}{9 + 3h} = \frac{-6}{9 + 3h} = \frac{-2}{3 + h}
\]
Appendix E (Continued)

E.8 Lauri: Quiz 2

\[ \frac{d}{dx} \left[ \frac{1}{x^2} \right] = \left( x^3 \right)^{-\frac{1}{3}} = x^{\frac{3}{2}} = 3 \cdot x^{-\frac{1}{2}} \]

\[ 3 \quad \frac{d}{dx} \left[ 3e^x \right] = (2e^x)' + (e^x)' = 3e^x \]

3) \[ A = 10 \cdot s^2 \]

\[ V = s^3 \quad \frac{2}{3} V = s \]

\[ = (\frac{2}{3} s^\frac{3}{2}) \]

\[ = 10 \cdot s^\frac{2}{3} - \frac{3}{3} + \frac{3}{3} - \frac{1}{3} \]

\[ = \frac{1}{3} s^\frac{2}{3} - \frac{3}{3} \]

\[ A = \frac{4}{3} \cdot s^{\frac{1}{3}} \]

4) 1) \[ f(x) = x^2 \]

\[ - \frac{1}{h} \left( (x+h)^2 - x^2 \right) \]

2) \[ f(x) = \frac{1}{x} \left( \frac{1}{x+h} - \frac{1}{x} \right) \]

3) \[ f(x) = \sin x \quad \frac{\sin(x+h) - \sin x}{h} \]
2. Setup the difference quotient:

\[ \frac{f(x) - f(a)}{x - a} \]

for each of the following functions:

(a) \( f(x) = x^2 \)

\[ \frac{x^2 - a^2}{x - a} \]

(b) \( f(x) = \frac{1}{x} \)

\[ \frac{\frac{1}{x} - \frac{1}{a}}{x - a} \]

(c) \( f(x) = \sin(x) \)

\[ \frac{\sin(x) - \sin(a)}{x - a} \]

3. Please evaluate \( f(2 + h) \) for each of the same functions above:

(a) \( f(x) = x^2 \)

\[ \frac{(2+h)^2 - (2)^2}{h} \]

(b) \( f(x) = \frac{1}{x} \)

\[ \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \]

(c) \( f(x) = \sin x \)

\[ \frac{\sin(2+h) - \sin(2)}{h} \]

4. (a) For the first function above, \( f(x) = x^2 \), use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \).

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} (6 + h) = 6 \]

(b) Find the tangent line to \( f(x) \) at \( x = 3 \) using the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \):

\[ \left(3, f(3)\right) \quad \text{m:} \quad f'(3) = \frac{6}{1} = 6 \]

\[ y - f(3) = 6(x - 3) \]

(c) For the second function above, \( f(x) = \frac{1}{x} \), use the limit definition of the derivative (not the power rule, though you may use it to check your work) to find \( f'(3) \).

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h \to 0} \frac{\frac{3 - 3}{3(3+h)}}{h} = \lim_{h \to 0} \frac{1}{3(3+h)} = \frac{1}{9} \]
Appendix E (Continued)

E.10 Drew: Quiz 2

I. \((10f(x) + 2g(x))'\)  
\(f'(2) = ?\)
\[10(-3) + 2(5)\]
\[-30 + 10 = \boxed{-20}\]

II. \(f \cdot g' + f' \cdot g\)
\((-3x^3) + (7x^5)\)
\[9 + 35 = \boxed{44}\]

III. \(\left[\frac{f(x)}{g(x)}\right]' = \frac{g \cdot f' - f \cdot g'}{g^2}\)
\(\frac{5(x^7) - (-3x^5)}{x^2}\)
\[\frac{35x^5 + 15x^2}{x^2} = \boxed{35x^3 + 15}\]

IV. \(\frac{f(x+h) - f(x)}{h}\)
1. \(f(x) = x^2\)
\(\frac{(x+h)^2 - x^2}{h}\)
2. \(f(x) = \frac{1}{x}\)
\(\frac{1}{x+h} - \frac{1}{x}\)
3. \(f(x) = \sin x\)
\(\frac{\sin(x+h) - \sin x}{h}\)
Appendix E (Continued)

E.11 Turner: Quizzes 1&2

Math 180 - Fall 2010
Quiz 3

Name:

Please write your answers on the back of this page.

1. Set up the difference quotient,

\[
\frac{f(x) - f(a)}{x - a},
\]

for each of the following functions:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin x \)

2. Evaluate \( f(2 + h) \) for each of the same functions above:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin x \)

3. Set up the difference quotient,

\[
\frac{f(x + h) - f(x)}{h}
\]

for each of the following functions:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sin x \)

4. Use your answer from 3a to find \( (x^2)' \) using the limit-definition of the derivative.

\[
\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x
\]
Appendix E (Continued)

E.12 Jo: Quizzes 1 & 2

Math 180 - Fall 2010
Quiz 3

Name:
Please write your answers on the back of this page.

1. Set up the difference quotient,

\[
\frac{f(x) - f(a)}{x - a},
\]

for each of the following functions:

(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

2. Evaluate \( f(2 + h) \) for each of the same functions above:

(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

3. Set up the difference quotient,

\[
\frac{f(x + h) - f(x)}{h},
\]

for each of the following functions:

(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

4. Use your answer from 3c to find \( (x^2)' \) using the limit-definition of the derivative.

\[
\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = \frac{h(x + h)}{h} = x + h = x
\]
Appendix E (Continued)

E.13 Nat: Quizzes 1 & 2

Math 180 - Fall 2010
Quiz 3

Name:

Please write your answers on the back of this page.

1. Set up the difference quotient,

\[ \frac{f(x) - f(a)}{x - a}, \quad \begin{align*}
\text{a) } \frac{x^2 - a^2}{x - a} &= f'(x) \\
\text{b) } \frac{1}{x} - \frac{1}{a} &= f'(x) \\
\text{c) } \frac{\sin x - \sin a}{x - a} &= f'(x)
\end{align*} \]

for each of the following functions:

(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

2. Evaluate \( f(2 + h) \) for each of the same functions above:

(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

3. Set up the difference quotient,

\[ \frac{f(x + h) - f(x)}{h} \]

for each of the following functions:

(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

\[ a) \ f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x \]

4. Use your answer from 3a to find \( (x^2)' \) using the limit-definition of the derivative.
2a. \( f(x) = f(2 + h) \) \[= f(2 + h) - f(2) \]
\[= \frac{\text{area}}{(2 + h) - 2} \]
\[= \frac{4h + 2h^2 - 2}{2 + h} \]
\[= \frac{4h}{h} \cdot \frac{2 + h}{2} \]
\[= 4 + h \cdot \frac{1}{2} \]
\[= 4 + h \cdot \frac{1}{2} \]

\[\frac{\text{area}}{h} = \frac{4 + h}{2} \]

2b. \( f(1) = \frac{1}{2} + \frac{1}{2} \)
\[= 4 + 2h \]
\[= \frac{2 + h}{2} \cdot 2 \]
\[= 4 + h \cdot \frac{1}{2} \]
\[= \frac{2 + h}{2} \cdot \frac{1}{2} \]
\[= \frac{4 + h}{2} \]

2c. \( f(x) = \sin x = f(2 + h) = \frac{\sin(2 + h) - \sin(2)}{2} \)
\[= \frac{\sin 2 + \sinh - \sin 2}{h} \]
\[= \frac{\sinh}{h} \]
Appendix E (Continued)

E.14 Sterne: Quizzes 1 & 2

1) \( \frac{f(x)-f(a)}{x-a} \)

2) Evaluate \( f(2, 3) \)

3) \( \frac{f(x+h)-f(x)}{h} \)

a) \( f(x) = x^2 \)

\[ \frac{x^2 - a^2}{x-a} \]

\[ \frac{(x+h)^2}{x^2} \]

\[ \frac{1}{h} \]

\[ \frac{1}{2 + h} \]

b) \( f(x) = \frac{1}{x} \)

\[ \frac{x}{x-h} \]

\[ \frac{1}{2 + h} \]

\[ \frac{1}{h} \]

\[ \frac{1}{h} \]

\[ \frac{1}{h} \]

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\[ \frac{1}{h} \]
E.15 Hudson: Quizzes 1 & 2

Math 180 - Fall 2010
Quiz 3

Name:
Please write your answers on the back of this page.

1. Set up the difference quotient,
\[ \frac{f(x) - f(a)}{x - a}, \]
for each of the following functions:
(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

2. Evaluate \( f(2 + h) \) for each of the same functions above:
(a) \( f(x) = x^2 \) \[ \frac{3n^2 + 4nh + n^2}{n} = \frac{n^2 (n + 2)}{n} = \frac{(n + 2) (n + 1)}{n} \]
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

3. Set up the difference quotient,
\[ \frac{f(x + h) - f(x)}{h}, \]
for each of the following functions:
(a) \( f(x) = x^2 \)
(b) \( f(x) = \frac{1}{x} \)
(c) \( f(x) = \sin x \)

4. Use your answer from 3a to find \( (x^2)' \) using the limit-definition of the derivative.
b) \( f(x) = \frac{1}{x} \quad \frac{1}{(2+\hbar)-(2+\hbar)} \cdot \frac{1}{h} = \frac{1}{h} \) 

DNE

0) \sin x
## Appendix F

### DATA ANALYSIS TABLES FROM QUALITATIVE PHASE

Figure 20: Number of Function Representations

<table>
<thead>
<tr>
<th>Number of Examples</th>
<th>Algebraic Expression</th>
<th>Graph</th>
<th>Name</th>
<th>Ordered Pairs</th>
<th>Table</th>
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<td></td>
</tr>
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</table>

Number of Examples: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Number of Examples: 1, 2, 3, 4, 5, 6, 7, 8, 9
Appendix F (Continued)

<table>
<thead>
<tr>
<th>Student</th>
<th>Unrelated to $f(x + h)$</th>
<th>Solve for $x$ or $h$</th>
<th>$h$+“equation” Transformation</th>
<th>Strength</th>
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<td>Lian (C)</td>
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<td>0</td>
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<tr>
<td>Ray (A)</td>
<td></td>
<td>✓</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Lauri (C)</td>
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<td>✓</td>
<td></td>
<td>−2</td>
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<tr>
<td>Drew (D)</td>
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<td>−2</td>
</tr>
<tr>
<td>Turner (C)</td>
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<td>✓</td>
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<tr>
<td>Jo (A)</td>
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<td>Nat (–)</td>
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<td>Sterne (F)</td>
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<tr>
<td>Hudson (D)</td>
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</table>
TABLE XXXVII: ASSESSMENT OF THE STRENGTH OF STUDENTS’ ACTION CONCEPTION OF $f(x + h)$

<table>
<thead>
<tr>
<th>Student</th>
<th>Explicitly evaluates with numbers</th>
<th>Plugs $x + h$ into $x$</th>
<th>External cue</th>
<th>Strength</th>
</tr>
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<tbody>
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<td>Lian (C)</td>
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<tr>
<td>Pierre (B)</td>
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</tr>
<tr>
<td>Ray (A)</td>
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<td></td>
<td>1</td>
</tr>
<tr>
<td>Lauri (C)</td>
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<td>0</td>
</tr>
<tr>
<td>Drew (D)</td>
<td>✓</td>
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</tr>
<tr>
<td>Turner (C)</td>
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</tr>
<tr>
<td>Jo (A)</td>
<td></td>
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<td>0</td>
</tr>
<tr>
<td>Nat (–)</td>
<td>✓</td>
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<tr>
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### TABLE XXXVIII: ASSESSMENT OF THE STRENGTH OF STUDENTS’ PROCESS CONCEPTION OF $f(x + h)$

<table>
<thead>
<tr>
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<tr>
<td>Ray (A)</td>
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<tr>
<td>Lauri (C)</td>
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<td>Drew (D)</td>
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<td>Turner (C)</td>
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### TABLE XXXIX: ASSESSMENT OF THE STRENGTH OF STUDENTS’ TOTALITY CONCEPTION OF $f(x + h)$

<table>
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<tr>
<th>Student</th>
<th>Horizontal shift</th>
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<th>Strength</th>
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<td>Ray (A)</td>
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<tr>
<td>Drew (D)</td>
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<td>Turner (C)</td>
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<td>Nat (–)</td>
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<tr>
<td>Sterne (F)</td>
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<tr>
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### TABLE XL: ASSESSMENT OF THE STRENGTH OF STUDENTS’ ACTION/PROCESS CONCEPTION OF FUNCTION

<table>
<thead>
<tr>
<th>Student</th>
<th>Missing Input/Output/ Transformation</th>
<th>Tied to Equation</th>
<th>Defined</th>
<th>Unique Outputs</th>
<th>Strength</th>
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<tbody>
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<td>✓</td>
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<td>Pierre (B)</td>
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<td>Ray (A)</td>
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</tr>
<tr>
<td>Sterne (F)</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>
TABLE XLI: ASSESSMENT OF THE STRENGTH OF STUDENTS’ PROCESS CONCEPTION OF FUNCTION

<table>
<thead>
<tr>
<th>Student</th>
<th>Coordinate</th>
<th>Solve for $x$ or $y$</th>
<th>Unique Outputs</th>
<th>No External Cue</th>
<th>Strength</th>
<th>Point-wise composition</th>
<th>Multiple representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>✓</td>
<td>✓ 1</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>✓</td>
<td>✓ 1</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ray (A)</td>
<td>✓</td>
<td>✓ 1</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>✓</td>
<td>✓ 1</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Drew (D)</td>
<td>✓</td>
<td>✓ 2</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Turner (C)</td>
<td>✓</td>
<td>✓ 2</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jo (A)</td>
<td>✓</td>
<td>✓ 2</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nat (–)</td>
<td>✓</td>
<td>✓ 1</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sterne (F)</td>
<td>✓</td>
<td>✓ 2</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>✓</td>
<td>✓ 1</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE XLII: ASSESSMENT OF THE STRENGTH OF STUDENTS’ OBJECT CONCEPTION OF FUNCTION

<table>
<thead>
<tr>
<th>Student</th>
<th>Action or Process on function</th>
<th>Uniform composition</th>
<th>Horizontal transformations</th>
<th>Strength</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>3</td>
</tr>
<tr>
<td>Ray (A)</td>
<td>✓</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>NA</td>
<td>✓</td>
<td>✓</td>
<td>2</td>
</tr>
<tr>
<td>Drew (D)</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Turner (C)</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Jo (A)</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Nat (−)</td>
<td>NA</td>
<td>✓</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Sterne (F)</td>
<td>✓</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>NA</td>
<td>✓</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
## TABLE XLIII: STUDENTS WHO DID AND DID NOT USE LANGUAGE “PLUG INTO f(x)” FOR DQ$_a$ EVALUATION

<table>
<thead>
<tr>
<th>Student</th>
<th>Used</th>
<th>Not used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>they give you $f(x)$ which is $x^2$ so I plugged in $x^2$ into $f(x)$ and then it will give us an $f(a)$.</td>
<td>$f(x) = \frac{1}{x}$, so I just plugged it into this general formula.</td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>the function equals $\sin(x)$, so I plug that into the $f(x)$ and then I subtracted $f(a)$</td>
<td></td>
</tr>
<tr>
<td>Ray (A)</td>
<td>the first one [circles $f(x)$ in the DQ$_a$], would be just $x^2$.</td>
<td></td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>$\text{The formula’s } f(x) - f(a) \frac{1}{x - a}$ so you’re taking $\frac{1}{x}$ and plugging into the formula.</td>
<td></td>
</tr>
<tr>
<td>Drew (D)</td>
<td>plugged in the $x^2$ into like the $f(x)$ and the $a^2$ into the…</td>
<td></td>
</tr>
<tr>
<td>Turner (C)</td>
<td>plug $f(x)$ into where the $f(x)$ would be so it would be just $\sin x - f(a)$ [all] over $x - a$.</td>
<td></td>
</tr>
<tr>
<td>Jo (A)</td>
<td>used the $f(x)$ function that they gave me and just um, plugged it into the equation that they gave me, the difference quotient.</td>
<td></td>
</tr>
<tr>
<td>Nat (–)</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Sterne (F)</td>
<td>plug that in for that [points to $f(x)$ in the difference quotient] and then $f(a)$.</td>
<td></td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>the equation that is given is $f(x) - f(a)$ over $x - a$ so it already gives me that $f(x)$ is $x^2$ so wherever there is an $f(x)$ I plugged in $x^2$.</td>
<td></td>
</tr>
</tbody>
</table>
### Appendix F (Continued)

**TABLE XLIV: STUDENTS WHO DID AND DID NOT USE LANGUAGE “PLUG INTO $f(x)$” FOR DQ$_{h}$ EVALUATION**

<table>
<thead>
<tr>
<th>Student</th>
<th>Used</th>
<th>Not Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>[13 second pause]. This would be $f(x+h)$ ’cuz you’re pluggin’ in, $x+h$ in the $x^2$. ’Cuz that’s what’s given. And then you plug in an $x^2$ into $f(x)$. #00:34:27.2</td>
<td>I just plugged—my function is $f(x+h)$—I just plugged $x+h$ into that $f(x)$ and then minus $f(x)$ which is just $\frac{1}{x}$. #00:27:34.8#</td>
</tr>
<tr>
<td>Pierre (B)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ray (A)</td>
<td>[starts doing DQ$_a$ at first, then corrects himself] this is the difference quotient, you have to add the $h$, $(x+h)^2 - x^2$ [all] over $h$. #00:23:39.9#</td>
<td></td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>$f(x)$ is equal to $x^2$. So you take $x^2$ [underlines $x^2$] and plug it into...all the $x$’s. No. [20 seconds pass]. #00:35:54.8#</td>
<td>you’re taking $x+h$ plugging it into this, and you’re getting $(x+h)^2$. The same thing you’re taking $x$ and you’re placing it in here. And you get $x^2$, minus $x^2$. And $h$ is just $h$. #00:41:31.1#</td>
</tr>
<tr>
<td>Drew (D)</td>
<td>I just plugged in like, the $f(x+h)$. I just plugged in the $x+h$ for $x$. For like the first part and then the second part, just $x$ into $x$. #00:47:24.8#</td>
<td></td>
</tr>
<tr>
<td>Turner (C)</td>
<td>So it’s just gonna be the $(x+h)^2$ and the $x^2$ over $h$. #00:24:48.9#</td>
<td></td>
</tr>
<tr>
<td>Jo (A)</td>
<td>plug in $x+h$ for $x$ in the function, so it would be $\frac{1}{(x+h)}$ minus $\frac{1}{x}$ all over $h$ [writes $\frac{1}{x+h}-\frac{1}{x}$]. #00:21:25.2#</td>
<td></td>
</tr>
<tr>
<td>Nat (–)</td>
<td>NA</td>
<td></td>
</tr>
<tr>
<td>Sterne (F)</td>
<td>you’re still looking at a quantity that’s $x+h$ squared and then just $x^2$ #00:38:23.7#</td>
<td></td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>$f(x+h)$ would equal $(x+h)^2$. It gives you $f(x)$ so I just subtracted $x^2$. #00:28:29.1#</td>
<td></td>
</tr>
</tbody>
</table>
Appendix F (Continued)

TABLE XLV: RELATIONSHIP BETWEEN LANGUAGE STUDENTS USED TO DESCRIBE DQ<sub>a</sub> AND DQ<sub>h</sub> EVALUATION

<table>
<thead>
<tr>
<th>Student</th>
<th>In Which DQ Student used “plug into f(x)” language</th>
<th>did not use</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>Both</td>
<td></td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>DQ&lt;sub&gt;a&lt;/sub&gt;</td>
<td>Both</td>
</tr>
<tr>
<td>Ray (A)</td>
<td>DQ&lt;sub&gt;h&lt;/sub&gt;</td>
<td>Both</td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>DQ&lt;sub&gt;h&lt;/sub&gt;</td>
<td>DQ&lt;sub&gt;a&lt;/sub&gt;</td>
</tr>
<tr>
<td>Drew (D)</td>
<td>DQ&lt;sub&gt;a&lt;/sub&gt;</td>
<td>DQ&lt;sub&gt;h&lt;/sub&gt;</td>
</tr>
<tr>
<td>Turner (C)</td>
<td>DQ&lt;sub&gt;a&lt;/sub&gt;</td>
<td>DQ&lt;sub&gt;h&lt;/sub&gt;</td>
</tr>
<tr>
<td>Jo (A)</td>
<td>NA</td>
<td>Both</td>
</tr>
<tr>
<td>Nat (–)</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Sterne (F)</td>
<td>Both</td>
<td></td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>Both</td>
<td></td>
</tr>
</tbody>
</table>

TABLE XLVI: TIME EACH STUDENT TOOK TO ANSWER FUNCTION COMPOSITION QUESTION 2 FROM INTERVIEW 2

<table>
<thead>
<tr>
<th>Student</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lian (C)</td>
<td>5:57</td>
</tr>
<tr>
<td>Pierre (B)</td>
<td>1:04</td>
</tr>
<tr>
<td>Ray (A)</td>
<td>0:06</td>
</tr>
<tr>
<td>Lauri (C)</td>
<td>0:22</td>
</tr>
<tr>
<td>Drew (D)</td>
<td>0:08</td>
</tr>
<tr>
<td>Turner (C)</td>
<td>0:10</td>
</tr>
<tr>
<td>Jo (A)</td>
<td>0:26</td>
</tr>
<tr>
<td>Nat (–)</td>
<td>9:34</td>
</tr>
<tr>
<td>Sterne (F)</td>
<td>0:25</td>
</tr>
<tr>
<td>Hudson (D)</td>
<td>0:26</td>
</tr>
</tbody>
</table>
CITED LITERATURE


Dubinsky, E., Arnon, I., and Weller, K.: Preservice teachers’ understanding of the relations between a fraction or integer and its decimal expansion: the case of 0.9 and 1. Manuscript accepted for publication, in press.


VITA

Education

2005–2012  **Doctor of Arts, Mathematics Education**, University of Illinois at Chicago (UIC), Chicago, IL.


1999–2003  **Bachelor of Arts, Mathematics Major, Computer Science Minor**, SUNY Binghamton University, Binghamton, NY.

Doctoral Thesis

**Title**: *Student Thinking of Function Composition and its Impact on the Ability to Set up the Difference Quotients of the Derivative*

**Advisor**: Dr. Janet Beissinger

**Description**: The purpose of this study is to investigate student cognitive processes involved in setting up the difference quotients and the associated errors. At the end of the study, a framework that aggregates criteria used (by past studies and this study) to assign student membership into a function conception category will be produced. Implications from this study can inform teaching practices by exposing students to expected errors.

Teaching and Research Experience

Pre-service and In-service Teachers


Work in Cryptoclub (a National Science Foundation funded project) to observe and record teacher and middle grade student interactions with material, develop computer games, manage and pilot mini-projects, teach in-service teacher workshops, analyze data, write research reports, examine effects of using Cryptoclub book to teach pre-service elementary teachers number theory.

- Introduction to Number Theory with Applications

Fall 2010  **Teaching Assistant**, Department of Curriculum and Instruction, UIC, Chicago, IL, (312) 413 - 0304.

Focused on meeting the needs of Chicago Public School students, particularly African American and Latino/a students, developed pre-service teachers’ confidence, co-led class discussions, held office hours.

- Mathematics for Elementary Teaching

Undergraduate Mathematics Teaching

Summer 2009  **Lecturer**, Department of Mathematics, Statistics, and Computer Science (MSCS), UIC, Chicago, IL, (312) 413-2175.

Designed, organized, and taught all lessons, developed mid-term and final exam, held office hours, about 30 students enrolled.

- Multi-Variable Calculus
Mathematics Instructor, Department of Mathematics, Columbia College, Chicago, IL, 312-369-7442.
Taught college mathematics covering algebra, geometry, trigonometry, and exponential functions to about 30 Art Majors, assign and grade homework, create and grade quizzes and midterm exams.
• College Math

Teaching Assistant, MSCS, UIC, Chicago, IL, (312) 413-2175.
Planned, organized, and led all discussion sessions, created and graded quizzes, graded exams, held tutoring hours for discussion.
• Linear Algebra (Fall 2008, Spring 2009)
• Introduction to Advanced Mathematics (Summer 2008)
• Quantitative Reasoning (Spring 2008)
• Calculus II (Fall 2004, Fall 2007)
• Introduction to Differential Equations (Summer 2007)
• Honors Calculus II (Spring 2005)
• Pre-Calculus (Fall 2003, Spring 2004)

Math Learning Center Supervisor, MSCS, UIC, Chicago, IL, (312) 413-2175.
Worked with the Associate Head of Instruction in recruitment of undergraduate tutors, created schedules for the center, supervised tutors, addressed concerns or question about the Math Learning Center.

Teaching Assistant Coordinator, MSCS, UIC, Chicago, IL, (312) 413-2175.
Worked with the Associate Head of Instruction to evaluate new TAs and give constructive criticism on how to improve teaching.

Middle Grade Students
Mathematics Instructor, Early Outreach Program, UIC, Chicago, IL, (312) 996-0979.
Instructor for a total of five advanced algebra classes for high-achieving minority and underrepresented 7th and 8th grade students (about 20 students in each class) from various Chicago Public Schools, designed, organized and taught all lessons, developed and implemented place-and inquiry-based assignments.

National Science Foundation Graduate STEM Fellow (GK-12), Scientists, Kids, and Teachers, UIC, Chicago, IL, 312-413-3904.
Worked with three middle grade teachers at a pre-dominantly African American Chicago Public School to introduce advanced mathematics to 7th and 8th graders, implemented inquiry learning, helped develop confidence within students, created mathematical projects that are place-based, attended workshops and seminars.

Posters and Talks
Speaker, “Student Thinking of Function Composition and its Impact on their Ability to Set up the Difference Quotients of the Derivative”, 15th Annual Conference on Research in Undergraduate Mathematics Education (RUME), Portland, OR.

Invited Speaker, “The Errors Calculus Students Make when Setting up the Difference Quotients of the Derivative’ and their Implications’, Northeastern Illinois University, Chicago, IL.

Speaker, “Student Thinking of Function Composition and its Impact on their Ability to Set up the Difference Quotients of the Derivative: A Preliminary Report”, UIC, Chicago, IL.

2006, 2007  Poster, “Fare or Unfare: Analyzing the 2006 Chicago Transit Authority Fare Increase”, Teaching for Social Justice Curriculum Fair, Chicago, IL.

Conferences Attended

2012  Joint Mathematics Meeting, Boston, MA.
2012  15th Annual Conference on RUME, Portland, OR, (Speaker).
2011  14th Annual Conference on RUME, Portland, OR.
2010  9th International Conference of the Learning Sciences, Chicago, IL.
2010  Transforming Research in Undergraduate STEM Education, Orono, ME.
2010  Chicago Symposia Series, Excellence in Teaching Mathematics and Science: Research and Practice, Chicago, IL.
2009  AMS Spring Central Sectional Meeting, Urbana-Champaign, IL.

Memberships

2011  Mathematical Association of America.
2003- present  American Mathematical Society.

Services

2011  Reviewer.

• Journal for Research in Mathematics Education
• Journal for Mathematical Behavior
• The Mathematics Educator

Computer Skills

PASW (SPSS), Atlas.ti, LaTex

Manuscripts in Progress

Tang, G., (In Progress), Framework for Membership in Student Conception Category.


References

Dr. Janet Beissinger, Research Associate Professor, MSCS & LSRI, UIC, Chicago, IL. (312) 413 - 2168, beissing@uic.edu

Dr. Danny Martin, Professor, Department of Curriculum and Instruction & MSCS, UIC, Chicago, IL. (312) 413 - 0304, dbmartin@uic.edu

Dr. Bonnie Saunders, Clinical Associate Professor, MSCS, UIC, Chicago, IL. (312) 413 - 1417, saunders@uic.edu