Knowledge Base of Mathematics Teacher Educators:

A Goals-Knowledge-Practice Approach

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THESIS

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Dedication

To my grandfather, Aleksander Matusovich Gritsovskiy
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<tr>
<td>CCSS</td>
<td>Common Core State Standards</td>
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<tr>
<td>CGI</td>
<td>Cognitively Guided Instruction</td>
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<tr>
<td>NCTM</td>
<td>National Council of Teachers of Mathematics</td>
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<td>MT</td>
<td>Mathematics teacher</td>
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<td>MTE</td>
<td>Mathematics teacher educator</td>
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<td>MTEE</td>
<td>Mathematics teacher educator educators</td>
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<td>PST</td>
<td>Pre-service teacher</td>
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SUMMARY

This study examines the work of four mathematics teacher educators (MTEs) while teaching mathematics content courses to pre-service elementary teachers. I use case study design to examine the knowledge of the four MTEs in my study, draw on in their teaching of the math content courses, and explain how these forms of knowledge relate to their goals and practices. Through a goals-knowledge-practice framework, developed as a result of my research, I examine their teaching, learning, and curricular goals and point to the particular knowledge and ideologies that inform those goals. I explain how the MTEs’ practices of instructional moves, activity structures, and decision-making relate to their goals and knowledge.

The findings from this study indicate that, in addition to developing mathematical content knowledge and skills, MTEs strive to develop confidence in their students’ mathematic abilities and positive dispositions toward learning and teaching mathematics. They draw mostly from their knowledge of teaching the content and other courses to pre-service teachers (PSTs) as opposed to their own educational experiences. The framework provides a lens for studying MTEs and a tool for professional development of MTEs.
CHAPTER 1

STATEMENT OF THE PROBLEM AND BACKGROUND

Statement of the Problem

Research on the topic of teacher education is being conducted by scholars across the globe. Often, this research is compiled in handbooks, reports, and volumes (Cochran-Smith & Zeichner, 2005; Cochran-Smith, Feiman-Nemser, McIntyre, & Demers, 2008; Darling-Hammond & Bransford, 2005; Houston, 1990; Murray, 1996; Sikula, Buttcry & Guyton, 1996). The Handbook of Research on Teacher Education: Enduring Questions in Changing Contexts (Cochran-Smith, et al, 2008), for example, presents many research-based answers to some of the essential questions in teacher education, such as What's the point?; What should teachers know?; Where should teachers be taught?; Who should teach?; Does difference make a difference?; How do people learn to teach?; Who’s in charge?; How do we know what we know?; and What good is teacher education? However, a section titled Who should teach teachers? is missing from this volume. Similarly, the authors of the chapters in Studying Teacher Education: The Report of the AERA Panel on Research and Teacher Education (Cochran-Smith & Zeichner, 2005) review the research in numerous areas of teacher education and give critical recommendations for future research on issues, such as the structure of teacher education programs, accountability practices in teacher education, and pedagogical approaches in teacher education. However, they neglect to review research studies about the practices and knowledge bases of teacher educators. They do not even point out the lack of research in this area of teacher education.

Recently, scholars have turned their attention to studying mathematics content courses for PSTs. Some of the questions being addressed include: how many and what kinds of mathematics content courses are offered for pre-service elementary mathematics teachers (McCory & Cannata, 2010); who teaches mathematics content courses to pre-service teachers (Masingila et al, 2012); and what should be the number, sequence, and topics to be included in mathematics content courses for PSTs (Conference Board of the Mathematical Sciences, 2001, 2012). Yet many other questions about the knowledge and practice of MTEs continue to remain unanswered: how do they know what to teach; how do they learn
how to teach teachers; how do they prepare to teach their courses; how does the research on teacher education inform their practice?

In mathematics specifically, the practice of teacher educating has beginning to receive increasing attention in the last two decades (Chauvot, 2008; Hiebert et al., 2003; Sztajn, 2011; Zaslavsky & Leikin, 2004). Sztajn (2011) attributes this growth of research in mathematics teacher education to the convening of the first meeting of the Association of Mathematics Teacher Educators (AMTE) in the United States in 1997. In 2012, together with the National Council of Teachers of Mathematics, AMTE began publishing the *Mathematics Teacher Educator* journal with the goal of supporting professional growth of mathematics teacher educators. This is an important milestone. Mathematics teacher educators have an important role in education. They mentor, teach, train, and offer support to pre- and in-service teachers, contributing to the development of their students’ knowledge and practices. My aim is to make a contribution to this emergent important work in the field of mathematics teacher education, by illuminating cases of MTEs’ practice, through analysis of their goals, knowledge, and ideologies. In addition to illuminating the cases, I am to describe the process of engaging in this work of researching the work of MTEs, and highlight the methods for engaging in this work.

Through a multiple-case study, I explore the relationship between MTEs’ goals, knowledge, ideologies, and practice as it manifests in the work of preparing for, instructing, and reflecting on mathematics content courses for pre-service elementary teachers. I collected data from four MTEs over the course of one semester in which they taught a mathematics content course for PSTs. With the help of multiple sources of data, such as interviews, observations, and documents, I was able to highlight key aspects of their goals, knowledge, ideologies, and practice. An additional contribution of my research study, is a framework, shown in Figure 1 below, which may be used as a lens for the study of the work of MTEs.
During my fifth year as a mathematics teacher I was asked to supervise a student teacher. That was
the first time I found myself in the role of a teacher educator. To help my student teacher develop into an
effective mathematics teacher, I modeled teaching for him, explained my instructional decisions,
answered questions about what he observed, helped him plan lessons, watched him as he taught, and gave
him feedback on his planning and performance. Many of my instructional strategies were shaped by the
practice of educators I admired. Consequently, at that time, I believed teaching by example was the best
approach to educating student teachers. Had I looked for literature to inform my work as a school-based

Personal motivation

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Figure 1. Goals-Knowledge-Practice Analytical Framework
teacher educator, I would have found that little has been written on the topic of the skills needed to be a school- or university-based teacher educator. Most research related to mathematical knowledge for teaching is concentrated in K–12 schools as it applies to mathematics teachers and not to mathematics teacher educators (MTEs) (Ducharme & Ducharme, 1996). Research about the work of teacher educators, or the knowledge and skills required for this work, has begun to become more abundant in the last few decades. In a volume on teacher education trends in the Netherlands (Willems, Stakenborg, & Veugelers, 2000), Korthagen (2000) writes that teacher educators have long been a neglected group. Although that is slowly changing as they become more involved in Dutch education projects. Similarly, teacher education research in the U.S. has largely neglected the topics of what teacher educators need to know and how they go about developing this knowledge (Cochran-Smith, 2003; Goodwin et al, 2014; Koster et al, 2005).

Nof and colleagues (1999) argue that one of the goals for doctoral dissertations for teacher educators should be to bolster up the knowledge base for teacher education. Guided by the research questions below, I intend to add to the knowledge base for teacher education with my study of MTEs’ goals, knowledge, and practice.

1) What goals for teaching and learning do MTEs develop in their teaching of content courses for pre-service elementary education students? How do these goals draw on particular forms of knowledge and inform classroom practices?

2) What practices do MTEs foster in their teaching of content courses for pre-service elementary education students? How are these practices related to their goals and knowledge?

3) Why do MTEs draw on particular forms of knowledge in their teaching of content courses for pre-service elementary education students? How are these forms of knowledge related to their goals and practices?

**Significance**

Much of the research on the knowledge needed to teach prospective mathematics teachers is done by practicing MTEs conducting research about their own practice (Chauvot, 2009; Marin, 2014; Nicol, 1997; Tzur, 2001). Although these studies serve as great contributions to the literature on the knowledge related
to the work of MTEs, the self-examination and self-reflection aspects of this work can interfere with analysis and investigation of alternative viewpoints and courses of action (Ball & Cohen, 1999). My study contributes to the research on the knowledge base of teacher educators by presenting another approach—an evidence-based understanding of the goals, knowledge, and practices used by MTEs as they work to develop pre-service mathematics teachers’ knowledge for teaching mathematics. Through careful documentation of the steps taken to study MTEs, I offer a description of methods for others planning on studying teacher educators. Additionally, I propose a framework as a lens through which to study MTEs practice by understanding and analyzing their goals, knowledge, and ideologies.

Key Findings

Mathematics pre-service teachers take content courses in order to develop their content knowledge. Through observing MTEs’ practice, conducting interviews with them, and analyzing their instructional materials, the unexpected goal of gaining confidence in doing and teaching mathematics surfaced. Although, not necessarily stated on a syllabus, lesson plan, or white board, each MTE had a deep desire to have students make a meaningful connection to mathematics and develop positive dispositions toward learning and teaching the discipline.

Knowledge was analyzed though the use of Cochran-Smith and Lytle’s (1999) conceptualizations of the relationships of knowledge and practice—knowledge-of-practice, knowledge-in-practice, and knowledge-for-practice. The data showed that most participants relied on knowledge-in-practice to plan, deliver, and reflect on their work. This was due to a lack of preparation for the explicit work of teacher educating. The participants used some of their knowledge-for-practice, especially if they received teacher training, which included exposure to theories of teaching and learning. Even those participants that held multiple mathematics degrees admitted that the way they were taught mathematics was not conducive for teaching it to pre-service teachers. They credited their experience teaching courses to PSTs and reflection on these experiences with developing the knowledge and skills needed for turning future teachers into effective practitioners.
The analysis of the relationships between the components of goals, knowledge, and practice contributed to the development of a framework, which can be used as a tool for studying the practice and professional development of MTEs.

**Organization of the Dissertation**

This dissertation is organized into eight chapters. Chapter 2 contains reviews of relevant literature. Specifically, I provide an overview of the literature on the preparation, professional development, and knowledge base of teachers and teacher educators, policy recommendations for teacher education, and literature pertaining to knowledge, goals, ideologies, and practice of teachers and teacher educators.

Chapter 3 describes the research design of the dissertation. In this chapter I explain how the study was conducted, describe the participants, and outline data collection.

In Chapter 4, I describe coding and analysis. This explains how analysis led to the development of the goals-knowledge-practice framework, and it explains the components of that framework.

Chapters 5–7 include the major findings of the study. Chapter 5 provides findings from the cross-case analysis of all of the research participants. Chapters 6 and 7 each focus on one participant at a time to provide a description of the participant’s goals, knowledge, ideologies, and practice.

In Chapter 8, I discuss limitations and implication of my work for the study and professional development of teacher educators, reflect on the analysis of the data, and propose directions for future research of the work of MTEs.
CHAPTER 2

REVIEW OF RELEVANT LITERATURE

I will begin with a review of the research with a discussion of the preparation and professional development of teacher educators as those are connected to their development of knowledge. Next, I will review the available literature about the knowledge base of teacher educators. Additionally, I will highlight some of the literature that framed my understanding of the components of this knowledge base—goals, knowledge, ideologies, and practice. Next, since the knowledge base of MTEs logically includes the understanding of the knowledge needed for teaching, I include an overview of the conceptualization of mathematical knowledge for teaching deemed necessary by various scholars for K–12 teachers. In addition to being a component of the MTEs’ knowledge base, these theoretical perspectives are a helpful way of looking at MTE knowledge and practice. Finally, I will review the literature on policy recommendations for implementing mathematics teacher education.

Preparation and Professional Development of Mathematics Teacher Educators

There is no certification process for becoming a mathematics teacher educator in the way there is for becoming a mathematics teacher. Nor is there a curriculum specifically for those who want to become teachers of teachers (Ball et al., 2009; Goodwin et al., 2014). Lunenberg (2002) describes the efforts of Dutch teacher educators to develop a curriculum for teacher educators, but just as with most research pertaining to their education and development, this curriculum is designed for beginning, not aspiring teacher educators. For those individuals, such as myself, who want to become MTEs, there is little research on becoming a mathematics teacher educator and few formal programs that deliver training for mathematics teacher educators (Goodwin et al., 2014; Zaslavsky & Leikin, 2004).

In 1990, Wilson conducted a systematic review of the literature about the recruitment and training of teacher educators in Europe. The data from his study showed that teacher educators lacked formal preparation in almost all European Union countries. In 2000, Buchberger and colleagues investigated the state of teacher educators’ preparation in Europe, and once more concluded that most teacher educators
come to the profession with no education or training in methodologies of teaching and learning suitable for adult learners (e.g. student teachers and in-service teachers).

In the U.S., the data from a study of 293 teacher educators, reflecting on their preparation/education for the work as teacher educators, revealed “happenstance in becoming engaged in teacher education . . . and lack of explicit development of teaching skills or pedagogies related to teacher educating” (Goodwin et al, 2014, p. 291). Even the few participants in this study who stated they sought out doctoral programs with the goal of becoming teacher educators felt that the programs’ curricula did not explicitly focus on developing knowledge and skills for teacher educating. It is not uncommon in U.S. universities to see doctoral students, who receive little preparation for working with teachers, placed as instructors of teacher education courses (Zeichner, 2005). Overall, not many mathematics teacher educators have had formal opportunities to learn to work with teachers. Most have been educated to be mathematicians or K–12 teachers and thus lack the knowledge relating to teachers as learners. While they have experience teaching mathematics that is beneficial for K–12 students, they may not have the skills to teach mathematics that is beneficial for teachers (Smith, 2003; Sztajn et al., 2006).

Although a background in school teaching is certainly helpful in educating prospective teachers, teacher education is not simply about the transmission of good teaching practices from instructors to prospective teachers. Therefore, expert teachers do not smoothly transition to become teacher educators without any specialized preparation (Murray & Male, 2005; Simon, 1994; Zeichner, 2005). Developing novice teachers into professionals who can make educated judgments in different teaching situations, such as employing a certain practice at a particular moment in a lesson, requires learning new skills (Ball, 1989; Bullough, 2005; Loughran & Berry, 2005; Ritter, 2007; Zeichner, 2005). Ball (1989) used the example of teaching third graders versus teaching prospective teachers in a methods course on how to teach third graders to explain the contrast between the two experiences:

The third-grade class does not, however, just mirror the methods class. It also contrasts with it, for the third graders do not bring the same baggage to the learning experience as do the prospective teachers. For example, the third graders depend on me less and on one another more . . . . They employ and invent a wide variety of strategies for solving problems: in their innocence, they are not bound to find "the formula." They have considerably more confidence in their solutions and speak with noticeably
more authority. They challenge one another freely, revise their ideas without apparent embarrassment, and use sophisticated mathematical language and ideas (e.g., conjectures, negative numbers) (p. 9).

As seen in this description, there are very specific differences between learning mathematics for the first time and learning to teach mathematics after being its student for at least a dozen years. Teaching teachers requires expertise in how to engage adult learners, help them unpack the mathematical baggage they may bring with them, and sometimes build confidence in their own mathematical abilities before they can start teaching mathematics to children.

With my study of MTEs, I plan to add to the emerging research on the backgrounds, knowledge, and competencies of teacher educators by investigating the ways in which their knowledge development and construction influences their goals, developing knowledge base, ideologies, and practice.

Currently, there are a limited number of theoretical frameworks within which the study of teacher educators’ knowledge and practice can be situated. However, there are frameworks for professional development of teachers that can be adapted for this purpose. The two main notions, which guide this portion of the literature review, are 1) becoming a teacher educator requires the development of knowledge specific to the profession of educating teachers, and 2) in order to gain a deep understanding of knowledge for teaching it is necessary to study the practice of those who educate teachers since professional knowledge manifests itself in practice. To further explore these notions, I first draw on Murray and Male’s (2005) and Simon’s (1994) conceptualizations of knowledge growth, which are necessary for one’s transition from a K–12 teacher to a teacher educator, then give examples of some of the professional development activities for mathematics teacher educators described in the literature.

My experiences teaching mathematics to high school students and working with new and pre-service mathematics teachers confirm what the literature states—becoming a teacher educator is not a simple process of transitioning from teaching content to K–12 students to teaching teachers (Goodwin, 2014; Ritter, 2007; Van Zoest, Moore & Stockero, 2006; Zeichner, 2005). Becoming an effective teacher educator necessitates an increase of learning at each level of education from that of a student to that of a teacher to that of a teacher educator (Murray & Male, 2005; Simon, 1994).
Murray and Male (2005) conceptualized the transition from teacher to teacher educator in the following way: school teachers are perceived to be *first-order practitioners* with K–12 schools as their place of work considered to be a *first-order setting*; and teacher educators are seen as *second-order practitioners* with higher education institutions as their place of work thought of as a *second-order setting*. By using interviews and questionnaires, Murray and Male collected data about the experiences of novice teacher educators as they made the transition from being *first-order practitioners* to *second-order practitioners*. The data they collected confirmed my beliefs that the transfer of knowledge from being a schoolteacher to becoming a teacher educator is not a smooth or straightforward process. Meaning, one does not go from being a good K–12 teacher to becoming an effective teacher educator without some kind of learning and transformation in between those two contexts. The teacher educators in Murray and Male’s (2005) study emphasized their needed to develop a new form of pedagogical knowledge in order to teach teachers. Similar to Smith’s (2003) findings, Murray & Male’s research exposed a need for wider and deeper knowledge of pedagogy and content, as well as an understanding of how to structure the learning process for adult learners, as some of the components of this new form of knowledge.

In their study, Murray and Male (2005) discovered that identity development as a second-order practitioner took a minimum of three years for their participants. Furthermore, they identified four factors that contributed to the tensions and challenges experienced by their participants and the length of time it took for them to transition from feeling like first-order to becoming second-order practitioners. The first factor was the absence of prior experiences relevant to the teaching of prospective teachers. This lack of pertinent experience related to university-based teacher educating, which created extra pressure for the novice teacher educators they studied. The second factor entailed gaining a new form of knowledge for teaching prospective teachers. Murray and Male explained that, though they have already developed an extensive knowledge base for teaching school children, the novice teacher educators were considered to be *experts become novice* because of their lack of experience in a second-order setting. The third factor was a lack of experience conducting research prior to becoming teacher educators. Research participants in Murray and Male’s study had limited experience with conducting research, but all were expected to
become researchers rather quickly after assuming roles as teacher educators. Lastly, Murray and Male cited a lack of induction and professional development at the higher education institutions as something that added to the challenges of the novice teacher educators in their study. They called on higher education institutions to provide teacher educators with a tailored induction that would allow them to analyze and understand the pedagogy of second-order work and how they may attain the knowledge and skills to function as second-order practitioners. A considerable portion of the development of new and extended knowledge necessary for educating teachers should happen before one becomes a teacher educator. My research study is designed to address all four factors identified by Murray and Male (2005)—gaining relevant experience by studying the practices of active teacher educators; understanding the new knowledge of second-order work and higher education teaching by observing teacher education courses; gaining experience with conducting research; and creating an experience that contributes to my professional development as a future teacher educator.

Simon (1994) presented a framework that showed the interconnection between different domains of teacher knowledge, such as knowledge of mathematics, knowledge of teaching, and knowledge needed for teaching others to teach mathematics. To illustrate the interconnections, Simon drew on the structure of the learning cycle offered by Karplus et al. (1977). The elements of the learning cycle are an exploration stage, a concept identification stage, and an application stage, which activates a new exploration stage thus making the process circular. Simon (1994) adapted the learning cycle model by characterizing the process as a spiral of two phases instead of three—the application stage of one iteration of the cycle becomes the exploration stage of the next iteration. Simon used his own classroom to describe Learning Cycle One: Learning Mathematics (Figure 2). In his classroom students explored a problem situation in groups and used this exploration to construct and identify concepts. Then the class was called together to discuss their thinking at which point the application stage was triggered by the class discussion or a question posed by the teacher, which led to a new cycle of ideas to be explored, conceptualized, and applied.
When applying this model to teacher development (see Figure 3), learning cycle one serves as the learning cycle of the teacher’s own mathematical experience as a student and is considered the exploration stage for learning about the nature of mathematics, or Learning Cycle Two: Developing Knowledge About Mathematics. The concept identification and application phases of learning about the nature of mathematics lead into the exploration of Learning Cycle Three: Developing Theories of Mathematical Learning. In the work Simon conducts with teachers, learning cycle one is continually incorporated into the exploration stages of cycles two and three for each specific mathematical concept. The exploration stage is followed by the concept identification stage, a discussion of the nature of mathematics and mathematics learning, which allows teachers to apply their notions about mathematics and theories about mathematics learning. This discussion triggers a new exploration stage in the succeeding learning cycles. Analogously, every application stage is incorporated into the exploration stage of the subsequent learning cycles.

Learning Cycle Four: Understanding Students’ Learning happens repeatedly for every specific mathematical concept, such as students’ understanding of ratio or polynomials. Learning Cycle Five:
Instructional Planning functions as an application phase for each of the four previous cycles since planning instructional activities for each new mathematical concept requires mathematical knowledge, knowledge of mathematical theories of how students learn, and understanding of students’ mathematical development as it relates to each specific mathematical concept. As teachers work on instructional planning with their colleagues they identify concepts and apply their ideas to the next cycle, Learning Cycle Six: Teaching. The actual structure of all six cycles together is not circular in shape (see Figure 3) but its design of repetition justifies its name—there is no end to be spoken of and the learning categories are applied throughout teachers’ life-long professional development.

This framework by Simon (1994) emphasizes the relationship between teachers’ mathematical learning, their learning about mathematical learning, and their learning about mathematical teaching. Simon stated that this recursive framework could be applied to the learning of MTEs by moving to the next level of recursion. He explained that just as mathematics teachers learn about teaching mathematics by studying students’ development and learning theories, so MTEs could also learn about learning how to teach mathematics teachers by being teacher education students, thus embedding new learning cycles for mathematics teacher learning into the existing learning cycle structure. The main point here is that “the learning demanded at each level (mathematics students, mathematics teachers, and mathematics teacher educators) increases exponentially” (Simon, 1994, p. 91). Simon argued that in contrast to what we have right now—little or no preparation for MTEs—the preparation for each level, from a mathematics student to a mathematics teacher to an MTE, should increase considerably. I agree, which is why instead of relying on my knowledge gained through the experience of teaching mathematics in a K–12 setting and teaching PSTs, I am conducting research to observe, analyze, and understand key aspects of the knowledge I will need to be an effective MTE.

Along the same line, Zaslavsky and Leikin (2004) looked at professional knowledge and capacities of MTEs as relative to mathematics teachers’ knowledge and capabilities. They conceptualized MTE knowledge as an extension of the knowledge needed by teachers to teach school mathematics. In order to illustrate the learning process of mathematics teachers and mathematics teacher educators, they adapted
Figure 3. Learning Cycles One Through Six. Simon, (1994, p. 88)
and expanded Jaworski’s (1994) teaching triad and Steinbring’s (1998) model of teaching and learning mathematics. By combining the two models into a three-layer model they offer a lens through which to examine the interplay between the learning of teacher and teacher educators. Jaworski’s (1994) triad of mathematics instruction (the management of learning, sensitivity to students, and the mathematical challenge) was used to form the first layer, a triad of mathematics teacher instruction (the management of mathematics teachers’ learning, sensitivity to mathematics teachers, and the challenging content for mathematics teachers). In the second layer, the teacher was identified as the student and the teacher educator as the teacher. In the third layer, the teacher educator was described as the student and the teacher educators’ educator as the teacher. This model was utilized in the professional development of five mathematics teacher educators to explain their professional growth as part of the community of practice.

To examine the side-by-side learning of teachers and teacher educators, Jaworski (2001, 2003) developed a model of co-learning, a process during which, by working together to investigate shared interests, each participant develops new forms of knowledge. Her work emphasizes communities of inquiries that conceptualize professional learning as a process that can accommodate the learning of multiple partners during a joint activity. Similarly, Zaslavsky, Chapman, and Leikin (2003) present a framework for professional growth of mathematics teachers, MTEs, and educators of MTEs. They explain that their model considers all parties engaged in mathematics education,

Students learn mathematics; mathematics teachers (MTs) facilitate students’ learning of mathematics, and also participate as learners in inservice programmes [sic]; MTEs facilitate the learning of MTs, yet also participate as learners in professional development programmes [sic] for MTEs; and mathematics teacher educator educators (MTEEs) facilitate the learning of MTEs (p. 880).

The model emphasizes the central role of tasks and reflection for each type of learner, specifies opportunities for direct and indirect learning, highlights the relationships and connections between the two, and outlines principles of the development of a community of practice which fosters life-long professional growth of all the participants involved.
Mathematical Knowledge for Teaching Teachers

Teacher educators are a diverse group of individuals. In mathematics education, MTEs can be faculty of undergraduate and graduate courses in education and mathematics departments who are engaged in the work of educating pre- and in-service teachers at community colleges, universities, and alternate teacher certification programs. Furthermore, MTEs can be school-based leaders, such as mentors, supervising teachers, or professional development facilitators providing support for practicing and pre-service teachers within K–12 school systems (Sztajn, Ball, & McMahon, 2006; Chauvot, 2009). This complex description makes it difficult to define the professional capabilities and knowledge of MTEs, and the literature on the topic is still emerging. A number of researchers have argued that the knowledge that MTEs draw on to prepare pre-service and in-service teachers is distinctively different from teacher knowledge (Rider & Lynch-Davis, 2006; Smith, 2003; Sztajn, Ball, & McMahon, 2006; Tamir, 1991).

One such perspective on the specialized nature of MTEs’ knowledge comes from Tamir, who in 1991 published an article called Professional and Personal Knowledge of Teachers and Teacher Educators. In the article, Tamir proposed a definition for personal-professional knowledge that guides the practice of teacher educators. Tamir stated that the term professional knowledge refers to “that body of knowledge and skills which is needed in order to function successfully in a particular profession. This knowledge is determined by two, commonly accepted procedures: (a) job or task analysis, and (b) consensus of the community of the people who are recognized professionals in a particular field” (pp. 263-264). These two descriptors of professional knowledge seem fairly straightforward. Yet there have been few studies of MTEs’ job analysis, and there is no consensus by the MTEs themselves on what constitutes the knowledge base for teaching prospective mathematics teachers, as evident by the lack of frameworks for mathematical knowledge for teaching teachers.

Tamir described two types of relationships between personal knowledge and professional knowledge—one having to do with acquisition of professional knowledge and the other having to do with application of professional knowledge. These relationships are an essential part of what makes each MTE’s knowledge base so unique. For example, two MTEs who have similar prior educational
experiences may develop different knowledge from these experiences since their interpretations are influenced by their preexisting cognitive structures. In turn, these differences are reflected in the application of instructional methods by the two MTEs. Even if the content they choose to teach in a mathematics content course for PSTs is the same (e.g. multiplication of fractions), each MTE may use a different instructional activity to serve a distinct purpose informed by the unique application of his or her own professional knowledge. Tamir proposed a blueprint for professional knowledge which consists of three categories based on the work of Lee Shulman (1986), subject matter knowledge, general pedagogical knowledge, subject-matter-specific pedagogical knowledge, and a fourth category, that Tamir calls teacher-education pedagogical knowledge. Teacher-education pedagogical knowledge is knowledge formed by the unique experiences of each teacher educator and that teacher educator’s reflection on those experiences. The particular ways in which all of the above-mentioned knowledge types are applied to the goals and purposes of a teacher educator’s work comprises a personal-professional knowledge and influences how that teacher educator uses his/her knowledge within his/her practice for his/her unique purposes.

Tamir’s (1991) characterization of professional knowledge is rather broad, which makes it at once useful and problematic. It is useful because Tamir’s identification of the broad categories of knowledge of teacher educators, with a unique perspective that the basic building blocks are highly dependent on one’s personal experiences and interpretation of those experiences, can be applied to many different disciplines and contexts. It is problematic because it lacks specific examples or further elaborations of each category. An aspiring teacher educator may not be able to identify each type of knowledge presented by Tamir within his or her own knowledge base. Certainly, this blueprint leaves room for researchers to build upon and dissect each category to make it more specific to their research aims.

Another examination of professional knowledge of teacher educators came from a research study conducted by Smith (2003) to identify the components of professional knowledge of teacher educators and ascertain how this knowledge is different from that of mathematics teachers, as communicated by teacher educators and novice teachers. Smith reviewed the relevant literature and asked teacher educators
from Israel and Sweden and novice teachers from Israel the following questions: "What does it mean to be a good teacher educator? (functions), What is the professional knowledge of teacher educators? (deriving from functions), and How does professional knowledge of teacher educators differ from that of teachers?" (2003, p. 5)

Smith (2003) learned that both teacher educators and novice teachers believed that teacher educators’ professional knowledge base should include knowledge about interpersonal communication, subject matter, pedagogy, didactics, and knowledge of recent and foundational literature on teacher education, as well as an understanding of how to apply this knowledge from research literature to their practice. In addition, both teacher educators and teachers should exhibit knowledge about assessment and knowledge of how to structure the learning process for the learner.

Although the participants of Smith’s (2003) study found numerous similarities in the professional knowledge of teachers and teacher educators, they nevertheless were able to point to a number of differences in the professional knowledge of the two groups. The first difference pertained to the depth and width of knowledge needed by teacher educators versus teachers. The participants of the study believed that teacher educators should develop a comprehensive knowledge of education and educational systems beyond that of teachers, who were perceived as having a need for the understanding of education and subject matter as it pertained only to their work context. Another difference between the knowledge of teachers and teacher educators was the level of transparency of thought. Teacher educators have to know how to teach and think aloud in order to communicate their professional knowledge to PSTs. Teachers have to do this to an extent when they explain how to do a problem to students, but they do not have to explain the structure of the lesson to their students, whereas teacher educators have to be explicit about their choice of representations and strategies and the order of activities in the lesson. According to Smith (2003) teacher educators’ abilities to be explicit about their thoughts and actions “requires a high-level of meta-cognition, it is verbalizing the reflection-in-action, the tacit part of professional knowledge in teaching” (p. 22). Lastly while both teacher educators and teachers need to know how to teach children,
the teacher educators’ knowledge must also include an understanding of how to structure the learning process and develop subject matter, pedagogy, and didactic competencies in adult learners.

Smith (2003) identifies some aspects of knowledge for teacher educators that are a matter of a deeper dive into the literature, including a comprehensive knowledge of education and educational systems, and other aspects that require a study of the literature on teacher education and the practices of effective teacher educators. Knowledge of how to make one’s thinking transparent in a way that is both accessible and rigorous is not easily picked up from reading about teacher education. In mathematics, for example, since prospective teachers come to mathematics content courses with a range of procedural and conceptual knowledge of mathematics, MTEs must accurately access this knowledge and then develop methods that are designed to move the PSTs’ understanding of mathematics in the direction of mathematical knowledge for teaching. The transparency of thought and approaches for teaching adult learners are of the essence here since mathematics content courses are not meant to be a review of mathematics learned in school but a deep study of mathematical content as it relates to teaching children. While observing content courses, I often heard PSTs question the instructor’s explanations with inquiries such as “I leaned to do this in a much easier way. Why are you making it so complicated?” or “Isn’t it easier to just teach them the rules?” In response to these questions, the MTEs must validate PSTs’ knowledge of the mathematical content but also provide clear justifications for why multiple conceptual models and explanations, which were not made clear to them as students, must be a part of their repertoire as teachers. Kari Smith’s (2003) study of the knowledge for teaching teachers and how it differs from the knowledge for teaching K–12 students provides several methods for future or practicing teacher educators to examine their practice in order to assess their knowledge base for components related to teaching teachers as supposed to teaching K–12 students.

In addition to the studies of MTEs described above, many MTEs conduct self-studies to examine their own knowledge as it relates to their work of teaching mathematics PSTs (Chapman, 2008; Chauvot, 2009; Nicol, 1997; Tzur, 2001). For example, Chauvot (2009) conducted a self-study by examining her journal entries, reading lists, and past syllabi to identify the types of knowledge she drew on in her job as a
mathematics teacher educator-researcher. Using the work of Lee Shulman (1986) as her framework and the artifacts of her practice as data, Chauvot created a knowledge map of her perceived knowledge required for being a mathematics teacher educator-researcher. Thus knowledge map consisted of subject matter content knowledge (SMCK), pedagogical content knowledge (PCK), curricular knowledge (CK), and knowledge of context (CK) for a mathematics teacher educator-researcher (MTE-R).

Chauvot applied the categories of knowledge in the map to her role as a math teacher educator (context 1) and as a mentor to doctoral students (context 2). However, for the purposes of this paper, I will only discuss the portion of her knowledge map indicating the knowledge needed for teaching courses to pre-service mathematics teachers (context 1). Chauvot drew on Shulman’s (1986) categories of knowledge to identify three types of knowledge that are essential to a mathematics teacher educator—subject matter content knowledge (SMCK), pedagogical content knowledge (PCK), and curricular knowledge (CK). In addition, Chauvot identified a type of knowledge she called knowledge of research and explained that she relies on her knowledge of research to teach her courses (SMCK), to select instructional methods and strategies (PCK), and to choose the materials she will use (CK). This demonstrates that knowledge of research informs her decisions in multiple ways and spans across the other three categories of knowledge identified. For Chauvot, subject matter content knowledge (SMCK) consists the same types of knowledge for MTEs as it does for mathematics teachers—knowledge of mathematics as a discipline, knowledge of mathematics for teaching, and knowledge of mathematics curricula. Pedagogical content knowledge (PCK) includes knowledge of the learner and the learning process as it applies to the specific subject matter one teaches, which includes knowing the common misconceptions students have and having appropriate instructional strategies in one’s repertoire to address these misconceptions. Both SMCK and PCK are essential to the planning of the course, delivery of the lessons, and assessment of students’ progress and ability. Lastly, Chauvot describes curricular knowledge (CK) as knowledge of available and appropriate curricular materials, understanding the development of topics in these materials, knowledge of how materials in other content areas relate to the one the MTE teaches, understanding mathematics as it applies to K–12 students and university students, and knowledge
of state, national, and accreditation standards. In the category of curricular knowledge, Chauvot identified a type of knowledge she calls *knowledge of human resources*. This knowledge includes knowledge of expert scholars in the field whose work the MTE can draw on in terms of specific topics, such as proportional reasoning or problem-solving. Knowledge of human resources is fundamental to designing the syllabus, assigning readings, and seeking out appropriate supplemental materials for students who need extra support or guidance.

Recently some scholars attempted to conceptualize teacher educators’ knowledge in the discipline of mathematics by studying the practice of other MTEs. Using the data from five iterations of a mathematics content course for elementary PSTs taught by four different MTEs, Castro Superfine and Li (2014) created a multimedia database that they used to explore the knowledge demands of these MTEs as they taught the content courses. It was of particular interest to them how this knowledge for teaching prospective mathematics teachers is different than the knowledge used by K–12 teachers to teach mathematics to their students. Castro Superfine and Li (2014) used the video from class sessions, audio from planning sessions, and a variety of artifacts, such as lesson slides, pictures of board work, and class handouts, to illustrate how three specific aspects of MTE knowledge are a different form of knowledge than that used by K–12 teachers. The first form of knowledge is the knowledge used to understand the nature of student errors. Castro Superfine and Li (2014) concluded that this knowledge is employed by both MTEs and K–12 mathematics teachers, but unlike K–12 teachers, MTEs use this knowledge to not only inform instruction but to suggest instructional moves that PSTs should use to analyze the nature of student errors. The second knowledge both MTEs and K–12 mathematics teachers utilize is the knowledge of elementary school mathematics curriculum, including knowledge of algorithms. The MTEs require this knowledge to not only perform and explain these algorithms, the same as K–12 teachers, but to connect the understanding of algorithms to teaching practice and uncover PSTs’ misunderstandings related to their conceptual understanding of algorithms. Finally, both K–12 mathematics teachers and MTEs benefit from knowledge about student learning, but the two groups utilize this knowledge in different ways. Whereas K–12 mathematics teachers use this knowledge to inform their teaching and
support student understanding of various mathematics concepts, MTEs need to make the purpose of studying K–12 students’ learning clear in their teaching of mathematical content to PSTs. In addition to uncovering new perspectives related to MTE knowledge and practice, Castro Superfine and Li provide an opportunity for others to uncover new knowledge demands of MTEs by doing something that many studies do not do—making explicit the process they used to analyze the practice of MTEs.

In volume four of the *International Handbook of Mathematics Teacher Education* (2008), Perks and Prestage (2008) look at three aspects of knowledge: practical wisdom, knowledge developed from being a classroom teacher, be it a K–12 or a college classroom; professional traditions, knowledge that is found in scholarly texts, academic journals, and curricular materials; and learner-knowledge, knowledge that the learner has developed as a result of his/her own schooling. In the same volume, Zaslavsky (2008) identifies the knowledge, personal traits, and responsibilities of MTEs. She states that both a mathematics educator and a mathematics teacher educator need to be “reflective, adaptive, flexible, open-minded, risk-taker, sensitive, confident, and enthusiastic about what he or she teaches” (p. 111). She asserts that the knowledge base of MTEs needs to include knowledge for teaching school mathematics and knowledge of ways to enhance teacher learning. The responsibilities of MTEs include engaging teachers in productive tasks, modeling certain behavior and traits that are important for teaching with productive tasks, and making explicit the reflection on the construction, delivery, and assessment of the tasks.

In addition to the work described above, several recent dissertations address the knowledge base of MTEs. One examines knowledge domains of MTEs and how these knowledge domains affect their activity as they design and implement pedagogical experiences in mathematics methods courses and professional development workshops for pre-service and in-service teachers (Zollinger, 2014). Another investigates the work of two MTEs, one teaches PSTs and the other designs and implements professional development experiences for in-service teachers, in order to identify task domains of their work and characterize the distinctive qualities of the domain of mathematical knowledge for teaching mathematics teachers (Zopf, 2010). A third analyzes the work of three MTEs teaching mathematics content courses to pre-service elementary teachers in order to develop a framework of mathematical knowledge required by
these MTEs to teach multiplication and division of fractions to PSTs (Olanoff, 2014). My study will contribute to this emerging literature in the field of mathematics teacher education by providing an evidence-based approach to the study of knowledge, skills, and sources that MTEs develop and use to teach mathematics content courses for elementary PSTs in different settings.

My research focuses on the practice of four MTEs in order to understand the key aspects of their knowledge base, thus adding to the literature on the knowledge base of teacher educators. However, since research in mathematics education has concentrated significantly more on the knowledge of K–12 teachers than on the knowledge of MTEs, I will now turn to literature that describes the knowledge base, practice, goals, and ideologies of mathematics teachers.

**Knowledge, Practice, Goals, and Ideologies**

To examine the relationships between the knowledge and practice of the MTEs in my research study I drew on Ball and Bass’ (2003), Ball and Cohen’s (1999), and Cochran-Smith and Lytle’s (1999) conceptualizations of knowledge and practice relationships. The quote below, about conceptualizing mathematical knowledge for teaching from practice, could just as easily be applied to the knowledge base of MTEs as well as mathematics teachers.

What do teachers do, and how does what they do demand mathematical reasoning, insight, understanding, and skill? We began to try to unearth the ways in which mathematics is entailed by its regular day-to-day, moment-to-moment demands. These analyses help to support the development of a practice-based theory of mathematical knowledge for teaching. We see this approach as a kind of “job analysis”, similar to analyses done of other mathematically intensive occupations, from nursing to engineering and physics (Hoyles, Noss, & Pozzi, 1997), to carpentry and waiting table. In this case, we ask:

- **What** mathematical knowledge is entailed in the work of teaching mathematics?
- **Where** and **how** is mathematical knowledge useful and used within the work of teaching mathematics? How is mathematical knowledge intertwined with other knowledge and sensibilities in the course of that work? (Ball & Bass, 2003, p. 5).

Using the work of Ball and Bass (2003) as a guide, I focused on the mathematics knowledge for teaching as it surfaced within the work of educating teachers, which included tasks such as evaluating different interpretations and models presented by PSTs, addressing their misconceptions or incomplete understanding of mathematical ideas, and creating examples to further their understanding of specific mathematical concepts. In addition, I collected data related to how the MTEs in my study developed
knowledge for teaching teachers and inquired about the sources of this knowledge. For example, if a student had partial understanding of the concept and was in need of additional explanation or models, I noted the MTEs’ moves in my field notes and, during the post-observation interview, asked about the knowledge required for these moves and the sources of their response. Sources of knowledge included educational and work experience, books and articles, conversations with colleagues, data from the research they conducted, learning from their students, and reflection in- and on-action.

Ball and Cohen (1999) explained the importance of teaching being examined through the study of practice by stating, “practice cannot be wholly equipped by some well-considered body of knowledge. Teaching occurs in particulars—particular students interacting with particular teachers over particular ideas in particular circumstances” (p. 10). Therefore, although I learned a lot from the interviews I conducted with MTEs, even they themselves may not always be aware of some aspects of their knowledge that emerged in practice brought about by a certain interaction or interpretation. Furthermore, new knowledge manifested during practice as a result of conjecturing and experimenting. Thus, MTEs were developing new knowledge as they taught, something I was able to capture in the observations and later ask the MTEs to elaborate on in the post-observation interview.

Ball and Cohen (1999) called for professional learning that emphasizes questions, analysis, and investigation to prepare professionals that are able to use knowledge to learn in and from practice. Practice-centered professional learning broadens and diversifies the knowledge base of teachers by enabling them to observe unfamiliar to them visions of teaching and learning. Ball and Cohen argued for making “systematic study and analysis of learning and teaching the core of professional education” (p. 16) and emphasized that for this to happen teachers and teacher educators must “cultivate the capacities to investigate teaching and learning, develop new claims on the basis of such investigation, and defend them with evidence and argument” (p. 16). Guided by the research literature related to the knowledge bases of mathematics teachers and MTEs, I collected and analyzed data in order to investigate the practice of MTEs and develop evidence-based claims about their knowledge bases.
Cochran-Smith and Lytle (1999) described three types of knowledge-practice relationships, which I applied to the practice of MTEs: knowledge-for-practice is the MTE’s formal knowledge of content, pedagogy, and various theories; knowledge-in-practice is the MTEs’ practical knowledge embedded in practice and the reflection on their practice; and knowledge-of-practice is generated when the MTEs are intentional about investigating their own practice. For my study, knowledge-in-practice played the largest role. In order to understand the key components of each MTE’s knowledge and its relationship to practice, I studied the videos of the MTEs teaching, as well as the interviews and documents, such as syllabi and lesson plans.

Knowledge-for-practice and knowledge-of-practice emerged in the conversations about their practice. For example, in the pre-observation interview, in order to learn how they used their knowledge of students and strategies, I asked each MTE why he/she chose the specific strategy for delivering instruction or the particular model to further student understanding. This knowledge also emerged in some of the post-observation interviews when I asked MTEs to explain why they chose a particular approach for addressing a student misconception or a particular structure for mathematical discourse.

In my review of the literature, I came across several different conceptualizations of knowledge, and I will use a number of them to depict the knowledge of my participants. Schoenfeld (2011) defines individuals’ knowledge as the information available to them for the execution of tasks or the accomplishment of their goals. He further describes the different kinds of knowledge: facts, procedural knowledge, conceptual knowledge, and heuristics. Perks and Prestage (2008) look at three aspects of knowledge: practical wisdom, knowledge developed from being a classroom teacher, be it in a K–12 or college classroom; professional traditions, knowledge that is found in scholarly texts, academic journals, and curricular materials; and learner-knowledge, knowledge that the learner develops as a result of his/her own schooling. Cochran-Smith and Lytle (1999) describe knowledge and practice in terms of images, which they define as “central common conceptions that seem symbolic of basic attitudes and orientations towards teaching and learning” (p. 253). This way of looking at knowledge paints a more complete
picture of the knowledge of MTEs, and is complementary to a philosophical claim that that knowledge is a subset of beliefs.

To connect knowledge and practice to goals, I adapted Schoenfeld’s (2000, 2011) work on decision-making in the classroom. Schoenfeld’s (2011) major claim is that what people do is a function of their resources. Decision making, for him, is a function of a person’s knowledge, available materials and resources, goals, and orientations (beliefs, values, and dispositions). By parsing lessons into small segments and assigning goals, resources, and orientations to these segments, Schoenfeld and his team are able to understand why certain decisions are made during instruction. He defines goals as “something that an individual wants to achieve” (p. 20) and elaborates that goals can be immediate, long term, local, or emerging. Schoenfeld (2011) conceptualizes knowledge as a resource that is “potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks” (p. 25). Finally, he explains that the term orientations is used as “an inclusive term encompassing a group of related terms, such as dispositions, beliefs, values, tastes, and preferences” (p. 29, emphasis in the original).

To describe the ideologies of my participants, I turned to the work of Ernest (1991), who drew from the work of Perry (1970), to describe ideologies in terms of mathematical philosophies. According to Perry (1970), a dualist perceives knowledge as right or wrong and knowing as obtaining knowledge from knowledge authorities; a multiplist can see a certain amount of uncertainty about knowledge and thus believes that authorities do not have to be all-knowing; and a relativist recognizes that knowledge is uncertain and knowing requires the knower to constantly evaluate the knowledge claims based on contextual evidence. These three perspectives fit together with components of the philosophy of mathematics since a number of people believe mathematics to be black or white, correct or incorrect, and having to do with rules and methods that have to be followed without deviation. Those who study mathematics appreciate that it is a discipline full of certainty, but also acknowledge that there may be multiple routes to getting the solution and that solution may be expressed in a variety of ways. They also know that although there are rules and methods in mathematics, those methods can and should be modeled, justified, and contextualized.
According to Ernest, the two main distinctions in the philosophy of mathematics are between absolutist and fallibilistic schools of thought. Absolutists state that mathematical knowledge is definite, although there are rational arguments for accepting or rejecting it. They believe that, even though there is a certainty to mathematics, that certainty can be reached by a multiplicity of approaches. Fallibilists, in turn take a relativistic approach by acknowledging the array of approaches and possible solutions to mathematical problems, but they call for mathematical knowledge to be evaluated within a principled framework. Ernest (1991) refers to these as public schools of thought. He also describes private schools of thought: a dualistic view of mathematics as a distinctive set of mathematical knowledge that is certified by authority; and a multiplistic view of mathematics as a distinctive set of mathematical knowledge that is not sanctioned by authority or any other source and is sometimes characterized as a view that mathematics is a collection of tools to be used whenever the situation calls for them.

Ernest further divides the ideologies into those suited for utilitarian social groups, such as industrial trainer; technological pragmatist; old humanist, purist social groups; progressive educator; and groups seeking social change, such as the public educator. Ernest (1991) says different social groups have different ideologies and interests that influence their perceptions of the purposes of education and mathematics. “The industrial trainers represent merchant classes and industrial managers. The old humanitarians represent educated and cultured classes, such as aristocracy and gentry,” (p. 125) Ernest (1991) explains. Although these descriptions make the groups seem outdated, they can easily be applied to today’s contexts. Industrial trainers and technological pragmatists see the development of knowledge as relevant to the attainment of better jobs. In today’s context, industrial trainers see vocation as the aim of education and are the masterminds behind vocational schools that advocate basic skills. They are most likely proponents of tracking students into a set course of study based on their perceived fixed abilities. Technological pragmatists, on the other hand, are proponents of STEM (Science, Technology, Engineering, and Mathematics) education and bringing more applied mathematics, such as statistics courses, into schools. The old humanists would disagree strongly with the previous two ideologies, as they believe that mathematics should be studied for its beauty and complexity. They do not concern
themselves with its application to trades or students’ lives. Finally, the progressive and public educators not only identify with a growth mindset and the ability of students to improve their skills overtime but also believe that it is the aim of educational structures to develop mathematical fluency and conceptual understanding in all students. Public educators take this view further by insisting that mathematics can be used as a tool for ensuring social justice and equality. Naturally, one’s ideology impacts one’s goals. Teacher educators draw on their ideology to decide what is important to learn, for whom, and in what manner or order.

Mathematical Knowledge for Teaching

By examining the standards for the evaluation of teachers in various states, Lee Shulman and his colleagues noticed that in the mid-1980s most states based the evaluation of a teachers’ practice almost entirely on teachers’ execution of procedures, rather than on teachers’ subject matter knowledge (Shulman, 1986). During this time, teachers’ evaluations were based on preparing and presenting lessons and understanding and managing youth and educational policies. Teachers were not evaluated on their knowledge of the subject they taught nor their ability to teach the subject to their students. Shulman proposed a framework for teacher knowledge consisting of “content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of educational ends, purposes, and values” (Shulman, 1987, p. 8).

Shulman’s (1986, 1987) framework of content knowledge in teaching, especially his notion of pedagogical content knowledge (PCK), has helped conceptualize the work of teachers in significant ways by setting in motion the wheels of research into how PCK relates to various disciplines. Since Shulman’s introduction of this topology, in the field of mathematics education, researchers such as Bromme (1994) and Ball, Thames, and Phelps (2008) have expanded and built upon his framework. Bromme (1994) drew on Shulman’s conceptualization to develop a topology of teachers’ professional knowledge consisting of five categories: “(a) knowledge about mathematics as a discipline; (b) knowledge about school mathematics; (c) the philosophy of school mathematics; (d) general pedagogical (and, by the way,
psychological) knowledge; and (e) subject-matter specific pedagogical knowledge” (p. 85). According to Bromme (1994), knowledge of mathematics as a discipline includes mathematical rules and procedures, modes of thinking, and methods learned by teachers while they are still school children. Knowledge about school mathematics includes knowing the mathematics specifically taught to children in grades K–12 and understanding the significance of the concepts taught in the context of school mathematics. Knowledge of the philosophy of school mathematics is a combination of knowledge about epistemological foundations of mathematics, mathematics learning, and human life and knowledge. Pedagogical knowledge is knowledge specific to the job of teaching, such as classroom management and communication with parents for the purposes of explaining and influencing student behavior. Finally, subject-matter specific pedagogical knowledge is pedagogical knowledge particular to the discipline being taught. It includes understanding of the sequence and method in which each topic taught should be presented to students, understanding of how to evaluate students’ knowledge for the purpose of informing future instruction, and knowing which ideas in the curriculum require more emphasis than others. For the most part, these categories mimic closely those of Shulman (1986)—pedagogical knowledge is included in Shulman’s framework, and knowledge of mathematics as a discipline and knowledge of school mathematics can be considered as part of content knowledge and curriculum knowledge. The exciting part of Bromme’s topology is his conceptualization of the philosophy of school mathematics knowledge. To illustrate how he came to this category Bromme described a study of two teachers: one teacher considered mathematics to be a logical system, and another teacher had a process-oriented conception of mathematics. Bromme underscores that these positions are more a perspective than they are beliefs, and this is why he prefers the word philosophy to describe them. I, in turn, would use the word disposition. In his example, Bromme described teachers’ dispositions towards mathematics. Dispositions can be examined as they relate to learning, teaching, and students. This is one of the most important, but often most overlooked, aspects of teacher knowledge that MTEs need to be aware of, and it was interesting to see how disposition was addressed in the courses of the MTEs in my research study.
Ball, Thames, and Phelps (2008) elaborated on the notion of PCK by creating a representation of *Mathematical Knowledge for Teaching* (Figure 4). Mathematical knowledge for teaching (MKT) is meant to describe “the mathematical knowledge needed to carry out the work of teaching mathematics” (p. 395). Ball, Thames, and Phelps (2008) view MKT as consisting of two areas of knowledge: subject matter knowledge and pedagogical content knowledge (PCK). Each area contains two domains, creating in total four domains of mathematical knowledge for teaching: common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). In addition to these four domains, Ball, Thames, and Phelps (2008) identify two types of mathematical knowledge for teaching that they hope to further explore: horizon content knowledge and knowledge of content and curriculum.

*Figure 4. Domains of Mathematical Knowledge for Teaching. Ball, Thames, & Phelps, (2008, p. 403)*
Included in the area of subject matter knowledge are two domains: common content knowledge (CCK) and specialized content knowledge (SCK). The first domain, CCK, is defined as “mathematical knowledge used in settings other than teaching” (Ball, Thames, & Phelps, 2008, p. 399), meaning that individuals who are not in the teaching profession would be able to perform such CCK tasks as solving a multiplication problem correctly or putting numbers in order from least to greatest. The second domain, SCK, is defined as “the mathematical knowledge and skill unique to teaching” (Ball, Thames, & Phelps, 2008, p. 400). Tasks that would fall into the domain of SCK do not typically have uses outside of teaching. They include connecting mathematical topics across grades, anticipating student errors, and interpreting and assessing student-invented algorithms.

In the area of PCK, Ball, Thames, and Phelps (2008) included two domains, knowledge of content and students (KCS) and knowledge of content and teaching (KCT). They described KCS as knowledge of mathematics in combination with knowledge of how students learn. Knowledge of this domain entails being able to assess the challenge level of tasks relative to the ability of students, anticipating common student misconceptions, and understanding of students’ thinking in relation to their development and the subject being taught. The second domain in the category of PCK, knowledge of content and teaching, combines the knowledge about mathematics with knowledge about teaching. This includes assessing the instructional advantages and drawbacks of different models and methods, deciding how to structure students’ mathematical discourse, and understanding how the design and sequence of instruction impacts student learning (Ball, Thames, & Phelps, 2008).

According to Bass (2005), the identification and characterization of SCK is one of the most important findings of Ball, Thames, and Phelps (2008) because it clearly illustrates mathematics teachers’ need of a specialized mathematical type of knowledge. Bass (2005) underscored that SCK is “not a diminutive subset of what mathematicians know” (p. 429) but a very specific type of mathematical knowledge that should be emphasized in the mathematics curriculum of content and methods courses for PSTs. PSTs who are mathematics concentrators do not need to take more mathematics courses, they need to take courses specifically designed for prospective mathematics teachers. This strong position by Bass made me think
that the same has to be true for MTEs, meaning that the knowledge that MTEs need to develop is not an advanced form of the knowledge that K–12 teachers develop but a specific mathematics knowledge for teaching teachers. This thought is consistent with my position that K–12 mathematics teachers and mathematicians cannot simply step into their roles as MTEs but require a preparation that is specific to their work with PSTs.

**Teachers’ Mathematics**

Ball, Thames, & Phelps (2008) theorized that mathematics PSTs need to have a specialized mathematical knowledge for teaching—a knowledge profoundly different from that of others studying mathematics for purposes other than teaching. Similarly, in Teachers’ Mathematics: A Collection of Content Deserving to Be a Field, Usiskin (2001) argued that mathematics content for teachers is not a weaker version of mathematics than mathematicians study, as it is sometimes perceived, but a particular collection of content deserving to be its own field, called teachers’ mathematics, and taught to PSTs as a sequence of two to three courses. Usiskin described teachers’ mathematics as consisting of three types of mathematical content knowledge he considers necessary for teachers: (1) mathematical generalizations and extension, (2) concept analysis, and (3) problem analysis. According to Usiskin, a mathematics teacher with a strong knowledge of mathematical generalizations and extension will know, for example, that there are multiple formulas for finding the area of a triangle, some using geometry and others that include trigonometric ratios. Developing this kind of knowledge is important because PSTs often start out viewing mathematics topics in a singular way, such as knowing one formula for the area of a triangle, taught to them during a geometry unit in elementary school. This narrow view limits PSTs’ understanding of mathematics, and in turn their ability to develop a deep understanding of mathematical content in their students, and stands in the way of them linking mathematics topics to one another.

Usiskin (2001) called the second kind of teachers’ mathematics concept analysis. He describes it as the ability to understand a mathematical concept with depth and breadth, including the reason for its origin, its different definitions, and multiple applications. Mathematics teachers should have a deep enough knowledge of mathematics to explain the history of different types of numbers to their students,
such as negative numbers, decimals, and fractions and how the need for each of these types of numbers arose. Knowing the answers to these and other mathematical mysteries is an important aspect of mathematics knowledge for teaching, especially when teaching younger children who are most curious about the interworking of the world around them. Teachers who have deep concept analysis knowledge can foster love, appreciation, and understanding of mathematics in children from a young age.

The third kind of teachers’ mathematics Usiskin (2001) identified is problem analysis. Performing problem analysis means approaching a problem from different points of entry and solving it in more than one way, as well as being able to look at the solution in relation to the problem, examine what has been done to reach it, and evaluate the need for each step in the process. Mathematics teachers use this knowledge to assess student misconceptions, design plans for intervention, and interpret non-traditional algorithms and unique strategies that children with different backgrounds bring to the classroom.

Instead of stating that teachers need specialized content knowledge, pedagogical content knowledge, and knowledge of students, Usiskin explained how each type of knowledge is applied to each area of teachers’ mathematical knowledge. The area of the triangle example captures several aspects of knowledge of extensions and generalizations. The teacher certainly needs strong content knowledge, but he/she also needs to know how to assess student understanding in order to determine which formulas for the area of a triangle the students will be able to understand. The teacher needs content-specific pedagogical knowledge in order to know how to structure the lesson to include the different representations of formulas for the area of a triangle. In addition, the teacher needs knowledge of learners and curriculum to facilitate student discourse surrounding the different representations of the area of triangles. Thus, while the content is at the center of Usiskin’s teachers’ mathematics, the goals of extending, generalizing, and performing content and problem analysis cannot be reached without knowledge that Usiskin, unfortunately, does not outline. Similar to what I attempted to do with knowledge of extensions and generalizations in the area of a triangle example, it would be beneficial to apply the work of task analysis to the other categories of teachers’ mathematical knowledge in order to uncover a more detailed knowledge base related to each category.
Profound Understanding of Fundamental Mathematics

To round out my review of mathematical knowledge for teaching, I turn to Ma’s (1999) conceptualization of profound understanding of fundamental mathematics (PUFM), characterized as a deep, broad, and thorough knowledge of fundamental mathematics. Ma explained that elementary mathematics is fundamental mathematics because it is comprised of arithmetic and primary geometry, the two branches of mathematics that are historically considered as foundational for all other branches of mathematics. Ma defined understanding a topic with depth as “connecting it with more conceptually powerful ideas of the subject” (1999, p. 121) and understanding a topic with breadth as connecting it with ideas of “similar or less conceptual power” (p. 121). For example, connecting addition of decimals to addition of whole numbers is an example of understanding with breadth, whereas connecting addition of both decimals and whole numbers to place value is an example of understanding with depth. Ma outlined four properties central to teaching and learning with profound understanding of fundamental mathematics: connectedness, multiple perspectives, basic ideas, and longitudinal coherence. Ma explained that teachers with PUFM constantly make connections between their conceptual and procedural understanding of mathematics, and the various areas of mathematical domains. According to Ma, teachers who develop PUFM are able to understand and offer different approaches to solving problems and explain advantages and disadvantages of each approach. This means, for example, knowing both the partitive and measurement models for division and being able to give an example and an explanation of a situation that would fit each model. Teachers who develop PUFM have a strong understanding of basic, but essential, ideas such as the relationship of each place value to the one next to it and the idea of equation. They believe that their focus on these basic ideas allows them to aid their students in understanding new, more complicated concepts. Finally, teachers who develop PUFM have a deep understanding of the entire elementary mathematics curriculum and teach in a way that sets the foundation for elementary mathematics topics that their students will encounter in future grade levels (Ma, 1999).

Earlier I mentioned that some elementary PSTs feel that they do not need mathematics for teaching courses since they have already studied mathematics in their K–12 educations and that MTEs are forced
to defend their teaching of the algorithms, which, at first glance, may look to PSTs like material they
already know. By outlining the properties crucial to teaching elementary mathematics effectively, in such
as way as to develop deep conceptual and procedural understanding of mathematical concepts compared
to surface level procedural understanding, Ma clearly illustrated that mathematics for teaching is “neither
a matter of ‘more of’ nor ‘to a greater depth than’ the knowledge expected of students” (Davis & Simmt,
2006, p. 294). Instead he has illustrated that it is “a collection of content deserving to be a field” (Usiskin,
2001, p. 86) of mathematics in its own right.

Knowledge of Students and Sociocultural Context

An understanding of how students’ socio-economic, racial, and cultural backgrounds factor into their
comprehension and learning of the subject is a vital component of mathematics teachers’ knowledge. I
find it helpful here to give an example from my own teaching. Two years into my teaching career, I
attained ESL and Bilingual (Russian) endorsements and was subsequently assigned to teach bilingual
mathematics classes. This posed a bit of a problem because all the students in these classes spoke
Spanish, while I did not. With language as a barrier, I turned to culturally responsive teaching (Gay,
2000) as a tool for teaching mathematics to students who were also learning English. Given my
experiences as an immigrant and an English-language learner, I was cognizant of the importance of using
language to empower students. When I was beginning to learn English alongside learning other subjects
in school the “English only” slogan made me angry and defeated. Teachers were quick to judge my
abilities (often unfairly), in subjects such as mathematics and science, based on my English proficiency
and not my knowledge of these subjects. With that in mind, and even though I did not speak Spanish, I
encouraged my students to give mathematical explanations in any language that they felt comfortable
with and insisted that we learn the mathematics vocabulary in both languages together (thankfully there
are many cognates between Spanish and English). Based on their verbal explanations and the diagrams
they drew, I was able to assess my students’ mathematical capabilities and provide them with the
language and mathematics support that they needed, but more importantly I was able to show my students
that what they bring to the classroom—their knowledge of the Spanish language—is a strength not a
deterrent to their learning of mathematics.

This is just one example of how attending to issues of racial, cultural, and linguistic diversity in the
classroom involves types of knowledge unlike those needed for teaching white or middle-class students.
In addition to knowing their students’ personalities, preferences, intellectual habits, misconceptions, and
interests, teachers must also have an understanding of the ways in which students’ personal, cultural, and
racial backgrounds influence their work in the mathematics classroom (Aguirre, Mayfield-Ingram, &
Martin, 2013; Ball & Forzani, 2011; Gutierrez, 2002; Gutstein, 2009; Martin, 2013; Tate, 1995). This
means, for example, understanding that students’ non-traditional algorithms for solving mathematics
problems may reflect their cultural or personal backgrounds and not their lack of knowledge. Many of my
former students, who came from Mexico, solved equations correctly by moving the numbers and
variables to different sides of the equation and combining like terms instead of applying the inverse to
each side of the equation, as is traditional in the U.S. Teacher awareness and sensitivity to issues of
language, culture, and race different from their own constitute a type of knowledge that needs to be
included in frameworks of mathematics knowledge for teaching alongside content and pedagogical
knowledge.

Context (where, when, and whom you teach), equity considerations (how well you understand who
your students are and how their race, culture, language, and other identities and social positions relate to
learning mathematics), and disposition towards learning and the learner (how you feel about what, how,
where, and whom you teach) are rarely attended to in mathematics content courses. However, teachers’
and the students’ racial, cultural, and socio-economic identities and dispositions impact the process of
learning mathematics in the classroom and should be considered as a type of knowledge that mathematics
teachers need in order to teach effectively and that MTEs should make an effort to address when teaching
PSTs (Aguirre, Mayfield-Ingram, & Martin, 2013; Gutierrez, 2013; Gutstein, 2005; Ladson-Billings,
should be incorporated into mathematics content courses as they have a direct relationship to mathematics
learning (Gutierrez, 2013; Martin, 2013). Before they enter the teaching profession, PSTs should be made to think about “who they will teach, their beliefs about these students, the social conditions in which they live and learn, and what role mathematics has played and can play, in the lives of their students” (Martin, 2013, p. 4). Regardless of the level of their content knowledge, teachers cannot teach students without making an attempt to get to know them (Martin & Mironchuk, 2010). Gutiérrez (2012, 2013) urges teacher educators to recognize that the study of equity must have a place in mathematics education for PSTs and that conceptualizations of the knowledge necessary for teaching must include political knowledge for teaching. Because mathematics in the Western world is associated with intelligence and status, mathematics teachers can serve as gatekeepers for certain groups of students, even if they are unaware or unwilling to serve in this role. This is why it is the job of teacher educators to make explicit the political nature of teaching mathematics to their students, including redefining mathematics and learning as political constructs that affect who learns, what is learned, and how it is learned in the mathematics classroom (Gutierrez, 2013). All of this is to say that a deep knowledge of content and pedagogy are necessary but not sufficient for being an effective mathematics teacher or teacher educator. “Who you are, whom you teach, and under what sociocultural, sociopolitical, and socio-structural conditions are equally important and will heavily influence how you teach and toward what ends” (Martin, 2013, p. 6).

**Policy Recommendations for Preparing Mathematics Teachers**

In this section, I will focus on recommendations for the preparation of mathematics teachers. These documents are designed to inform the work of mathematicians and mathematics teacher educators. In my review, I will focus on the Conference Board of the Mathematical Sciences (CBMS) (2001, 2012) documents since the authors explicitly state that their recommendations are aligned with earlier National Council of Teachers of Mathematics (NCTM) recommendations and are geared to those individuals who teach mathematics to pre-service teachers.

Motivated by the publication of the NCTM *Principles and Standards for School Mathematics* (NCTM, 2000), CBMS released its first volume of *The Mathematical Education of Teachers* (MET I) in
2001. The authors of MET I believed that this publication was vital to “strengthening school mathematics instruction, to make it both more demanding and more effective for all students” (p. 5). The report, which contains recommendations for the preparation of pre- and in-service elementary, middle grades, and high school teachers, is designed to be a resource for anyone involved in mathematics teacher preparation, with a primary audience of mathematicians and mathematics education faculty of colleges and universities.

The authors of MET I credited Ball and Ma with revealing the truly complex nature of school mathematics and the profound understanding it takes to teach this area of mathematics effectively. Therefore, MET I focuses on two important themes: “(i) the intellectual substance in school mathematics; and (ii) the special nature of the mathematical knowledge needed for teaching” (p. xi). The authors stated that the “intellectual content in school mathematics instruction” (p.3) requires specialized mathematics knowledge for teaching that MTEs must foster in PSTs. This is an important statement, since many aspiring teachers believe, due to their own schooling, that elementary level mathematics is easy to teach because it is a collection of disconnected facts, definitions, and procedures to be memorized and not a complex and challenging discipline (CBMS, 2001).

The authors of MET I presented three groups of general recommendation for what are believed to be necessary improvements to teacher education as a whole, followed by detailed mathematical content recommendations organized by grade-level bands. The general recommendations, which apply to all grade levels in MET I, are broken up into three categories: curriculum and instruction for mathematics PSTs, collaboration between mathematics departments and other individuals or parties concerned with the education of mathematics teachers, and policy that supports strengthening mathematics teaching and learning. The purpose of the general recommendations was to provide departments of mathematics, departments of education, and MTEs with a set of core principles and recommendations to inform the design of their programs, courses, and instruction. For example, one of the recommendations states that PSTs must have an understanding of the fundamental principles of school mathematics and a thorough knowledge of mathematics several grades past and prior to those that they expect to teach. Another recommendation states that the coursework for PSTs (for elementary school PSTs, MET I recommends a
minimum of nine semester hours with concentration on the fundamental ideas of elementary school mathematics) should focus on fundamental ideas that develop PSTs’ reasoning skills and an understanding of how mathematical ideas are connected to one another. It is recommended that the coursework of elementary PSTs should be built in a coherent way, which emphasizes that mathematical theory and procedures are connected. To help PSTs see that students will engage with mathematics in different ways, the authors of MET I recommended that instructors model different styles of presenting new ideas, such as visualizing, proving, and modeling. Some PSTs might come to a mathematics course feeling angst and fear towards mathematics, because the way it was taught to them in school did not appeal to their learning style. Therefore, it is important that the teacher education courses PSTs take contain opportunities for them to experience numerous approaches to engaging students in mathematics. By witnessing their own understanding deepen as a result of these approaches, PSTs will learn that when they become certified teachers they also need to have a repertoire of methods for engaging their students with mathematics (CBMS, 2001).

Within the more detailed recommendations, MET I is organized in two parts. Chapters in part one outline recommendations for the mathematical content of teacher education courses by (a) grade clusters: elementary school, middle grades, and high school, and (b) mathematical domains: number and operations, algebra and functions, geometry and measurement, and data analysis, statistics, and probability, using the categories from Principles and Standards for School Mathematics (NCTM, 2000). Chapters 3–6, in the first part, provide a summary of the key topics for each grade cluster and technology-related teacher preparation in most of the mathematical domains (specifically number and operations, algebra, geometry, and data). In part two, Chapters 7–9 outline the key skills and knowledge PSTs in each grade cluster need to attain in each mathematical domain and include detailed examples of classroom scenes, in the form of vignettes, that show how the teaching and learning of these skills unfolds in the classroom.

In 2012, in response to the many changes that have occurred in education since 2001, such as the introduction of the Common Core State Standards (CCSS) (Common Core State Standards Initiative,
2010), demographic changes in the makeup of schoolteachers, an increase in the amount of teachers being prepared via non-tradition education programs, a new accreditation organization, new requirements for professional development, and increased involvement of mathematicians and statisticians in the mathematical education of pre- and in-service teachers, CBMS released *The Mathematical Education of Teachers II* (MET II) (CBMS, 2012). Understanding the mathematics topics prospective teachers need to know, requires a “clear vision” (CBMS, 2012, p. xii) of the mathematics that they will teach once they are certified teachers, and the authors of MET II see the CCSS as that “clear vision” of the mathematics that PSTs need to know in order to be effective mathematics teachers. The authors of MET II offered revised recommendations for the preparation and professional development of teachers of mathematics.

Although most of the recommendations in MET II are consistent with those in MET I, there are some changes. One major change is the recommendation that mathematics course requirements be raised from nine semester hours to twelve for prospective elementary school teachers, from 21 semester hours to 24 for prospective middle-grades teachers, and from one six-hour capstone course to three courses with a focus on high school mathematics from an advanced standpoint for pre-service high school teachers of mathematics. It is clear that the authors of MET II are supporters of CCSS—the new content recommendations contain many references to and examples from the CCSS, including recommended activities tied directly to the CCSS’ Standards for Mathematical Practice (SMP) to be used in teacher education courses and professional development sessions. The SMP is a portion of the CCSS that describes the different skills that the authors of CCSS believe mathematics teachers should develop in their students while teaching them mathematics. For example, teachers who want their students to develop the skills described in the SMP may ask their students to make sense of problems and persevere in solving them, reason abstractly and quantitatively, and look for and make use of structure in mathematics. These practices were derived from two important prior documents: the Process Standards from NCTM’s Principles and Standards of School Mathematics (2000) and the strands of mathematical proficiency identified in the National Research Council’s report Adding It Up (2001). The Process Standards specifically discuss problem solving, reasoning and proof, communication, representation, and
connections. Adding it up (NRC, 2001) discusses “adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy)” (CBMS, 2012, p. 83).

The strengths of MET I and II (CBMS, 2001, 2012) lie in the detailed recommendations of mathematical content to be covered in courses for PSTs, but these details are not unique to these two documents and can be gauged from such publications as the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010), Professional Standards for Teaching Mathematics (NCTM, 19991), or Principles and Standards for School Mathematics, (NCTM, 2000). A critical analysis highlights both the strengths and the weaknesses of MET I and II. (CBMS, 2001, 2012).

Based on my experiences with the MTEs in my study, I question the level of fidelity and adherence to these policy documents. The MET I (CBMS, 2001) authors recommend a three course sequence of at least 9 semester hours of mathematics coursework for elementary PSTs (raised to 12 hours in MET II (CBMS, 2012)), but I did not find any course sequence available that was longer than 2 courses ranging from 2.5 hours per week to 5 hours per week. Furthermore, none of my research participants referred to either of the MET volumes as a resource they drew on, nor did they ever come up in my teacher or doctoral education.

Though not as prevalent in MET II (CBMS, 2012), the authors of MET I (CBMS, 2001) consistently use deficit-oriented thinking and language when describing the knowledge and abilities of PSTs claiming that prospective teachers “struggle to gain minimal understanding of basic concepts” (p. 14), “have been less than successful mathematics students” (p. 23), and “Instructors teaching teachers for the first time will occasionally feel dismay, or even shock . . . ” (p. 56). This last warning to the instructors by MET I (CBMS, 2001) is followed up with an explanation that “these teachers’ mathematical backgrounds are consequences of systemic rather than personal failings, and it is essential that, recalling this, instructors work to maintain the necessary stance of interest, generosity, and respect” (p. 57). The fact that PSTs are
spoken of in such a deficient way, without research references or data to back up these claims, and that
the authors feel the need to remind instructors to maintain interest and respect is problematic for many
reasons. The authors describe a vicious cycle of poor K–12 education, underprepared college students,
and an inadequate college education that they claim exists, but neglect to explain how MTEs escape this
cycle and become qualified to teach PSTs. Although the authors make recommendations for the education
of PSTs, there seems to be an assumption that mathematicians are magically able to develop all these
skills in the PSTs without any mention of the kind of education and professional development they need
for the task of educating teachers. The absence of data or research-based literature to reinforce deficit-
oriented claims about PSTs makes it empty rhetoric, but not all readers will see it as such. Some
prospective or current instructors of mathematics content courses for PSTs may turn to this document and,
upon getting the impression that most of their students are incompetent and underprepared, set low
expectations for their courses, thus further adding to the problem of teachers whom MET I describes as
“ill-equipped to offer a different, more thoughtful kind of mathematics instruction to their students”
(CBMS, 2001, p. 17).

Both MET I and II (2001, 2012) contain inconsistencies between statements such as “the mathematics
knowledge needed for teaching is quite different from that required by college students pursuing other
mathematics-related professions” and “mathematicians are particularly qualified to teach mathematics in
a connected, sense-making way that teachers need” (CBMS, 2001, p. xi). The questions I have after
reading these statements are: How do mathematicians, who belong to the “other mathematics-related
profession,” develop this specialized knowledge? Doesn’t it make more sense that mathematics education
faculty are more likely to have this knowledge as a result of their teaching-specific degrees? These
questions are not addressed by either volume, and the inconsistency is further perpetuated by numerous
mentions of mathematics from a teacher’s perspective and mathematicians teaching courses for teachers
as two vital components for success in mathematics content courses.

Finally, just as with the frameworks for teaching mathematics, there is a glaring absence of a
discussion about equity considerations in mathematics content courses for teachers, which leads me to
conclude that MET I and II (CBMS, 2001, 2012) not only serve as partial guides for what to teach PSTs but fail to address the more important question of how to teach them.
CHAPTER 3
RESEARCH DESIGN

In *Qualitative Inquiry and Research Design: Choosing Among Five Approaches*, Creswell (2007) presented five commonly used qualitative approaches to inquiry in social, behavioral, and health science research: narrative research, phenomenology, grounded theory, ethnography, and case study. Creswell (2007) stated that the focus of case study is “developing an in-depth description and analysis of a case or multiple cases” (p. 78) and that the type of problem best suited for case study design is one of “providing an in-depth understanding of a case or cases” (p. 78).

With this research of four mathematics teacher educators who teach content courses to pre-service elementary education students, I studied and analyzed the goals they develop, the knowledge they draw on, and the practices they foster, as well as the relationship of these constructs of goals, knowledge, and practice to each other.

I drew on multiple sources of data, including observations, interviews, and documents in service to within-case and cross-case analyses. I selected case study as a research design because of the focus, problem type, and data collection forms of my research study. Additionally, I chose it because my main focus is the knowledge and goals for teaching prospective mathematics teachers not the experience of a specific MTE as in narrative research nor the meaning of several MTEs’ lived experiences as in phenomenology.

Stake (2005) identifies three types of case studies. If the researcher’s purpose is to better understand a particular case, such as a specific person or a particular organization, and not to understand a phenomenon or build a theory, then the researcher should perform an *intrinsic case study*. If, however, the researcher wants to gain insight into a particular issue or refine a theory, he or she will conduct an *instrumental case study* by selecting and examining a case that helps advance the understanding of the issue of interest. In contrast with intrinsic case study where the case itself is of the foremost importance, in instrumental case study, the topic of study is the most important component and the case is selected to develop a deeper understanding of the issue.
For this study, I used a collective case study design. In a collective case study, one topic of study is selected, but multiple cases are studied to understand the topic. By conducting a collective case study, I was able to gain insights into the knowledge of MTEs by analyzing the data within each case as well as across all cases. A cross-case analysis provides snapshots of how the constructs of goals, knowledge, and practice relate to each other and of the various shifts that occur between the constructs. With respect to the research questions guiding the study, a cross-case analysis provides a more general understanding of how the goals, knowledge, and ideologies of my participants impacted their practice. The individual case studies provide an in-depth examination of the knowledge, goals, and practice of each MTE in a group of MTEs who teach mathematics content courses to elementary PSTs.

Focus on Mathematics Content Courses

For my study, I wanted to concentrate on a specific discipline. I chose mathematics because of my prior experience as a mathematics teacher. Because I am curious about how the knowledge for teaching mathematics at the K–12 level differs from the knowledge needed to teach teachers how to teach mathematics, I chose to study instructors of content courses. Although, depending on the instructor or the institution, issues of pedagogy, equity, and assessment may be incorporated into mathematics content courses, the main focus is on mathematics content. Some of my research participants have admitted that they occasionally feel they have crossed the line into teaching methods in addition to content. However, I still feel that a mathematics content course for teachers can be used as a site for understanding the knowledge needed to teach mathematics to prospective elementary school teachers. It may seem that instructors of content courses for mathematics PSTs need only have a strong background in mathematics and some teaching experience, but I believe that there is more to the knowledge base of MTEs who teach content courses than a strong knowledge of mathematics and an understanding of how to teach it to K–12 students. For this reason, I conducted a study of their practices to learn about other key elements of the knowledge needed to teach elementary PSTs how to teach mathematics.
Focus on the Education of Elementary Pre-Service Teachers

Unlike secondary PSTs who concentrate in a particular subject (such as mathematics), elementary PSTs must know how to teach mathematics regardless of which subject they choose as a concentration. Presumably, because secondary PSTs are familiar with higher-level mathematics, they understand its complexity. Alternatively, for some elementary PSTs, their experience learning mathematics while they were students has made them believe that mathematics is full of boring rules and procedures to be memorized. As a result, many elementary PSTs dislike mathematics and doubt its intellectual character as well as their own mathematical abilities (Ball, 1989; 1990; Conference Board of the Mathematical Sciences, 2001, 2012). Other elementary PSTs may have a very procedural understanding of mathematics and therefore may believe teaching mathematics to elementary school students is as easy as having them memorize rules and procedures. MTEs must not only develop elementary PSTs’ conceptual and procedural understanding for teaching mathematics to young children but also have to convey to them the conceptually challenging and interesting nature of mathematics. This will result in PSTs feeling that they can be good at mathematics once they understand why the rules and procedures, which they may have only memorized up until now, actually work. Increasing elementary PSTs’ conceptual understanding and confidence will improve their ability to teach mathematics to their prospective students. This added level of complexity of the elementary MTEs’ knowledge of students, teaching, and learning compelled me to study the practice of MTEs in hopes of learning more about the special nature of teaching mathematics to pre-service elementary mathematics teachers.

Selection of Participants

In order to compose cases that offer the most opportunity to learn (Stake, 2005), one goal for the recruitment of participants for my research study was to select MTEs who self-identified as experienced teacher educators. Another goal I had for participant selection was to recruit MTEs who teach mathematics content courses to pre-service elementary school teachers in diverse settings. Because I believe it would be beneficial to compare the knowledge bases of MTEs in terms of the environment in which they teach, I tried to select theoretically diverse cases (Yin, 2006) that include two-year and four-
year, private and public, urban and suburban institutions, as well as courses that include graduate and undergraduate students. All of the participants had to be instructors of mathematics content courses for pre-service elementary teachers at higher education institutions. For the purpose of this study, I considered all individuals who teach mathematics to pre-service teachers at the university level to be mathematics teacher educators (MTEs). I looked on the university websites and reached out to the individuals who were scheduled to teach mathematics content courses for elementary PSTs, since that is my focus, during the winter/spring 2015 quarters or semesters. In all, four instructors who self-identified as MTEs responded and agreed to participate in the study. All of the institutions where my participants taught had a two-course sequence of mathematics content courses for elementary PSTs. Two of the research participants taught the first course in the sequence, and the other two taught the second course. I studied the practice of these four MTEs over the course of one semester or quarter during the winter and spring of 2015. I believe that focusing on a small, but deliberately selected, number of research participants allowed for depth of data in each case with a possibility of individual variations across cases.

The first participant, Alec, is an instructor in the mathematics department at a public community college. He has previously taught mathematics courses to college students at other universities. After being asked to teach the mathematics content courses for teachers, he developed a deep passion for teacher educating and took it upon himself to choose goals, objectives, and curricular materials that he felt best met the needs of future teachers. Although he still teaches other mathematics courses at the college, he admitted to enjoying the mathematics content courses for teaching best of all and feeling that he has the most impact with these particular courses.

The second participant, Sal, is an instructor in the mathematics department at a private four-year institution. He continues to teach other mathematics courses in the department and teaches these courses to continue the legacy of his mentor, who was extremely influential in the fields of mathematics and teacher education. Sal is the only one of my participants who did not use a textbook for the course, explaining that he found the mathematics-for-teachers textbooks to not be interesting or deep enough.
Fay is an instructor in the education department at a private four-year institution. She received her undergraduate degree in mathematics from the institution where she currently teaches. She teaches several methods and foundations courses for elementary and secondary PSTs. In an effort to better understand her students’ growth and progression from course to course and use her observations to inform her teaching, Fay petitioned the mathematics department to teach mathematics content courses for teachers.

Tina, like Fay, is an instructor in the education department at a private four-year institution. She came to her work as a teacher educator after many years as a high school mathematics teacher and several years as the director of a national grassroots-based mathematics intervention project, which worked to provide poor minority students with deep mathematical understanding through creative, hands-on approaches to learning. Unlike Fay, she did not have to petition the mathematics department because, as she stated, most faculty in the mathematics department do not want to teach the content courses for teachers. She, along with several other faculty members, trade off teaching these courses. Additionally, Tina teaches mathematics methods courses for elementary and secondary PSTs and student teaching seminars. Table I, below, summarizes the characteristics of the four research participants.

It is important to note that the participants in my study were self-selecting, meaning that they chose to participate in the research. Self-selection may have affected the results as these, most likely, highly motivated individuals had motivations and purposed for becoming participants in a research study on the knowledge base of mathematics teacher educators. Although I did not ask the participants about their motivation, I can deduce some of it from the data collected. For examples, Fay expressed confidence and thoughtfulness in designing her course materials, activities, and structures. Presumably then, her motivation may have been to illuminate her own practice as an experienced teacher educator. Tina expressed, during several interviews, that she had planned to record and analyze her class sessions in order to examine different aspects of her teaching and her students’ learning. Therefore, she was looking forward to doing this work as part of my research study. Alec was very reflective and often thought about his goals and practices while writing class summaries. Participating in my study gave him an opportunity to share these reflective thoughts with another person. The bulk of Sal’s reflection was done at the end of
the course. After the last session and before the final interview, I asked each participant to document their reflection on their teaching and their students’ learning as they graded the final assessments. Sal made some important realizations in terms of his students’ individual development of new knowledge and skills, which may have been influenced by my request to use the grading of exams as a reflective activity.

Table I. Overview of Research Participants

<table>
<thead>
<tr>
<th>MTE</th>
<th>MTE’s title</th>
<th>MTE’s education</th>
<th>Course title</th>
<th>Institution type</th>
<th>Years of teaching experience</th>
<th>Years of teacher educating experience</th>
</tr>
</thead>
</table>
| Alec   | Faculty Member in the Mathematics Department | B.S. in Mathematics  
M.S. in Mathematics  
Masters work in Education  
Ph.D. work in Mathematics Education | Mathematics for Elementary Teachers 1  
(undergraduate) | Urban 2-year Community College | 13                            | 9                       |
| Sal    | Senior Lecturer in Mathematics, Assistant Director of Undergraduate Studies | B.S. in Mathematics  
B.A. in French  
M.S. in Mathematics  
Ph.D. in Mathematics | Elements of Mathematics Instruction 2  
(graduate) | Urban Private 4-year University | 16                            | 5                       |
| Fay    | Assistant Professor of Secondary Education/Instructional Technology | B.A. in Mathematics and Psychology  
M.Ed. in Mathematics Education  
Ph.D. in Mathematics Education | Mathematics for Elementary School Teachers  
(Course 1 of 2)  
(undergraduate) | Suburban Private 4-year University | 21                            | 15                      |
| Tina   | Associate Chair & Associate Professor in Elementary Mathematics Education | B.S. in Education (major in Mathematics; minor in Afro-American Studies)  
M.A. in Inner City Studies  
M.A. in Secondary Mathematics  
Ed.D. in Specialized Educational Development | Mathematics for Elementary Teachers II  
(undergraduate) | Urban Private 4-year University | 42                            | 21                      |
Data Collection

To investigate the knowledge base of MTEs, I collected multiple sources of evidence to document various components of each of their practice (Yin, 2009). According to Yin (2009), there are six common sources of evidence: documents, archival records, interviews, direct observations, participant observations, and physical artifacts. To build a case study of each MTE’s knowledge, I collected evidence from the following sources: documents (e.g., course syllabi, lesson plans, and instructional handouts), direct observations, and interviews. The quality of the case study and the strength of the findings are improved greatly when they are supported by more than one source of evidence with multiple sources of information providing evidence for the same phenomenon (Yin, 2009). The concluding findings about what constitutes the knowledge base of mathematics teacher educators whose practices I examined became better defined because I used several sources of data (e.g. observations, interviews, and documents). Table II provides an overall summary of data collection for my research study.

Observations

Although I could learn a lot about the topics taught to elementary PSTs from studying the textbooks for mathematics content courses for teachers or content recommendations for mathematics education of teachers, it is not possible to grasp the full picture of teaching from studying the curricular and scholarly materials alone.

The purpose of the observations was to study the practice of MTEs in their natural environment of teaching courses to PSTs in order to learn about their goals and the key components of knowledge the MTEs draw on while working with PSTs in mathematics content courses. Therefore, for each of the courses selected by the MTEs for me to observe, I observed 6–12 out of the average 10–15 sessions taught in a quarter, and 10–20 out of the average 20–30 class sessions taught in a semester. I videotaped the lessons using two stationary cameras. One of the cameras was positioned to best see the MTE, and the other camera was placed in the back of the classroom to give an overview of the proceedings of the lesson and to best capture activity structures and interactions between the MTE and PSTs. Additionally, I asked each MTE to wear a clip-on recording device in order to capture his/her interactions with students when
Table II. Description and Purposes of Data Sources

<table>
<thead>
<tr>
<th>Source</th>
<th>Description</th>
<th>Purpose/Uses</th>
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<tbody>
<tr>
<td>Initial interview</td>
<td>45-60 minute in-depth interview with each participant: audio data, written response documents, my notes</td>
<td>To learn about MTEs’ educational and teaching experiences, and their goals and beliefs on the topics of teaching, learning, mathematics, and teacher education; their knowledge of theories about mathematics teaching and learning and reform, and the sources of this knowledge.</td>
</tr>
<tr>
<td>Classroom observations</td>
<td>6-15 classroom observations (about 15-20 hours) of each participant: video data, audio data, field notes</td>
<td>To study the practice of MTEs in order to learn about the key components of their knowledge and its relationship to their practice.</td>
</tr>
<tr>
<td>Pre- and post-observation interviews</td>
<td>10-15 minute semi-structured pre and post-lesson interviews of each participants: audio data, my notes</td>
<td>To determine the goals and plans for each class session and learn about the knowledge and sources of this knowledge used to prepare materials and instructional strategies for the class session (pre). To discuss if the plans were enacted as planned, reflect on choices of materials and strategies, and examine the moves and decisions made during the lesson (post).</td>
</tr>
<tr>
<td>Final interview</td>
<td>45-60 minute interview with each participant: audio data, written response documents, my notes</td>
<td>To reflect on the goals, knowledge, practices, and strategies used, and ask questions formulated based on the observations and previous interviews.</td>
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the MTE was not in view of the camera. Since the purpose of my observations was to study the practice of MTEs, I did not participate in any of the classroom discussions. I took field notes of each class session on my computer from a place determined by each MTE as least disruptive for observation.

**Interviews**

I designed an initial interview tool that I used for an in-depth interview with each of my research participants. The purpose of this interview was to learn about the background, education (including graduate and undergraduate portions, majors, and degrees), and teaching experience of each MTE and how these background educational and professional experiences influenced his/her instruction. I asked each MTE about his/her notions of what constitutes an effective mathematics teacher educator and what the MTE believes is the main job of a mathematics teacher educator. Next, I asked each MTE about his/her experiences with teaching teacher education courses in general and mathematics content courses in particular. I asked each research participant about the organization, teaching materials, and goals.
associated with the course that he/she has chosen for me to observe. Each MTE was also asked about the
theories and perspectives that he/she draws on to inform his/her philosophy about learning and teaching
mathematics and why he/she draws on those particular theories. Finally, I asked each MTE to reflect on
the differences and similarities between his/her knowledge base as a mathematics teacher educator and
the knowledge base that each MTE is developing in his/her students who will go on to become
mathematics teachers.

Before each observation, I conducted a short semi-structured pre-observation interview with each
MTE to determine the goals and plans for each class session and learn about the knowledge and the
sources used to prepare materials and instructional strategies for each class session. During this interview,
I asked each MTE to describe his/her teaching and learning goals, as well as the mathematical ideas of the
lesson and the success criteria he/she uses to assess the attainment of these goals for her/himself and the
PSTs. In addition, I asked about the knowledge necessary for developing these mathematical ideas in
PSTs and the sources (such as textbooks, articles, or human resources) that informed the instructional
strategies and materials selected to achieve the goals of the lesson. I also asked each MTE about
anticipated challenges in the lesson for the instructor and the students, and how the MTE plans to address
these challenges.

Following each observation, I conducted a short semi-structured post-observation interview to
discuss if the plans were enacted as planned and get answers to any questions I might have come up with
while observing. During this interview, I asked each MTE if he/she considered the goals for the lesson
met and if he/she thought that the instructional materials and strategies chosen were selected
appropriately. Next, I asked each MTE if anything unexpected or surprising happened during the lesson,
and if so, how it was addressed. Finally, I asked each MTE about changes to the plan or delivery of the
lesson for future semesters and the rationale behind these changes.

At the end of the semester, I conducted one final in-depth interview with each of the MTEs. During
this interview, I asked each MTE to reflect on his/her goals for the semester and students’ performance
toward meeting those goals. I also asked about possible revisions to instructional materials and strategies
and the reasons for these changes. Additionally, I asked questions I had formulated based on the
observations and previous interviews. In addition to asking the MTEs to reflect on the semester, the final
interview also included questions about the MTEs’ research and professional development pursuits and
how these experiences shape their knowledge base for teaching teachers. The interview protocols for the
initial, pre- and post-observation, and final interviews are included in Appendices A, B, and C,
respectively.

Documents

The final source of evidence I examined was documentation I collected pertaining to the course as a
whole (e.g. syllabi) and specific class sessions (e.g. lesson plans and handouts). I collected several
different types of documents from the four research participants, including lesson plans, post-lesson
summaries, classroom handouts, presentation slides, and syllabi. These documents had multiple purposes.
They served as supplemental evidence from interviews or observations and were used to demonstrate
each of the MTE’s goals and beliefs regarding the session’s objectives. The documents functioned as
another source of evidence contributing new information or adding information to the data collected from
observations and interviews.
CHAPTER 4

ANAYLSIS, CODING, AND EMERGENT FRAMEWORK

I collected data from multiple sources, including interviews, observations, and course documents. I analyzed the data using thematic analysis (Braun & Clarke, 2006). Thematic analysis is a process of examining and reexamining the data in increasingly refined ways in order to generate a rich interpretation. This process involves six phases: “familiarizing yourself with your data, generating initial codes, searching for themes, reviewing themes, defining and naming themes, and producing the report” (Braun & Clarke, 2006, p. 87).

I began the first phase of thematic analysis, familiarizing myself with the data, well before all of the data was collected. During data collection, I took field notes, videotaped the lessons, and audiotaped interviews. Afterward, I reviewed my field notes, audio and video data, and wrote memos to document my observations about what I saw and heard and brainstorm potential codes. For example, upon hearing, one of my research participants, Tina, describe creative ways of engaging students with activities from the textbook as “lifting it off the book” (Post-observation interview, May 21, 2015), I wrote a memo about her actual words being a potential in-vivo code. Once all the data was collected, I transcribed it, during which time I wrote more memos about codes, definitions, questions, and initial analysis.

In phase two, generating further initial codes, during open coding I used descriptive and in-vivo codes (Miles, Huberman, & Saldaña, 2014) to generate over 200 codes. I used the method of writing each code on a piece of paper and sorting them into groups to organize the codes and engage with phase three, searching for and generating potential themes. Phase three was done in multiples stages. I first grouped all the codes into a multitude of categories based on the different stages of teaching (i.e. pre, post, and during), and different processes (i.e. goal setting, instructional moves, identifying and addressing challenges, assessment, planning, etc.). Then, in phase four, to review the themes, I revisited my research questions, and grouped the codes into the categories of goals, knowledge, practice, and other. At this point, I did an initial round of phase five and defined some of my themes, knowing that some of the definitions and categories would be revisited and possibly defined more clearly at a later stage. I defined
goals as guidelines that explain what the teacher educator wants to achieve, practice as the actual work that teacher educators do, and professional knowledge as a body of knowledge and skills, which is needed in order to function successfully in a profession.

Once the broad groups of codes were created, I categorized the codes within each group further, often drawing on the literature I have read about goals, knowledge, and practice of educators, in order to define and name themes. For example, all statements that had to do with theories of learning and teaching were placed under the category of *knowledge-for-practice*, and all statements having to do with experience and practical knowledge were placed in the category of *knowledge-in-practice* (Cochran-Smith & Lytle’s, 1999). Since MTEs’ knowledge is complex, there was sometimes overlap between a number of categories. For example, the following sentence from a class summary, written by Alec (February 25, 2015) was coded under video, alternative methods and algorithms, mathematical knowledge for teaching, and multiple representations: “Providing alternative methods to solve these problems, or being clear in why the algorithms work instead of just showing students how to do the algorithm, will typically help to develop a students’ conceptual understanding” (class summaries, February 25, 2015). Other segments of the data fit more clearly into the initial categories, as shown in Table III.

Three main categories emerged, with two to five subcategories in each, as shown in Figure 5 below. In addition, I had a category called “Other” containing such codes as motivation, service, curiosity, and taking ownership. At first, I thought the theme “beliefs” would be a good fit for this category, but later, I decided on “Ideology” as the name for the category since the codes encompassed so much more than just beliefs. I placed this category inside the knowledge theme. The participants’ beliefs about students being open-minded and curious about mathematics or their insistence that learning is a life-long process felt a part of something larger than a belief, such as a world view that by being open-minded and curious they are acknowledging multiple perspectives. The theme of “multiple perspectives” and “the individuality of learning,” in turn, felt more like a philosophy than a belief. As defined by Ernest (1991), an ideology is a “value-rich philosophy or world-view, a broad interlocking system of ideas and beliefs” (p. 111), and Schoenfeld describes a person’s knowledge as “those understandings that the individual takes to be true
<table>
<thead>
<tr>
<th>Category &amp; definition</th>
<th>Sub-category</th>
<th>Code Samples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goals are guidelines that explain what the teacher educator wants to achieve.</td>
<td>Mathematical content goals</td>
<td>My goals are to help students to see why the long division algorithm works by looking at the place value, visuals, and the scaffold method. I want them to know the inner workings of addition, subtraction, multiplication, division, and pretty much any rational number within reason.</td>
</tr>
<tr>
<td>Mathematical knowledge for teaching goals</td>
<td>We just did it also this past week, addition and subtraction in different bases. I'm juxtaposing that with addition in base 10 and trying to create the drawings for them. We want to see if there are any inventive ways other than the algorithms and the models for subtraction of fractions.</td>
<td></td>
</tr>
<tr>
<td>Socially-oriented goals</td>
<td>I could tell them time and time again, “We have to be open to perspectives.” Develop an understanding for the need to help students rely more on themselves to determine whether something is mathematically correct.</td>
<td></td>
</tr>
<tr>
<td>Knowledge is the body of knowledge and skills needed in order to function successfully in a profession.</td>
<td>Knowledge-for-practice</td>
<td>In terms of representations, I have to thank my prior officemate getting into the nitty-gritty of representing fractions and the arithmetic. We would have conversations in the office about it and play with the snap cubes and do that, so he was kind of teaching me how to teach that. I remember taking the plane from New York to Chicago and reading <em>Other People's Children</em> and reading Kozol.</td>
</tr>
<tr>
<td>Knowledge-in-practice</td>
<td>As I was teaching, I would discover all these things and get really excited about it. Think about all the stuff that I do now that I've learned in these last eight years about this content. It's remarkable. It's a whole body of knowledge that I didn't have before. Prior to teaching these courses, I never thought about why multiplication in base 10 works as it does. Therefore, the knowledge I draw on for this lesson is very MFET-specific, acquired through teaching and reading about multiplication in base 10 (and other bases).</td>
<td></td>
</tr>
<tr>
<td>Ideologies</td>
<td>I don't think many of my students hold the perception of women being worse than men with respect to STEM, but that false perception still exists. I want them to be able to easily dismiss that. The same goes for perceptions along socioeconomic lines. Effective teachers reflect critically on the moral, political, social, and economic dimensions of education.</td>
<td></td>
</tr>
<tr>
<td>Practice is the work that teacher educators do.</td>
<td>Activity structures</td>
<td>I wonder if with this class, the same issues that keep recurring, more structured group activity and really mixing them up. I think that’s something I need to do. That's another way I'd open up. We have the problem, and you're solving the problem. Then, we see how students solved it.</td>
</tr>
<tr>
<td>Instructional moves</td>
<td>I used the girl in the middle as my litmus test because, well, she's in the middle, first of all. Second of all, she has had negative reactions to things and been quiet at times. When she's engaged I take her as a proxy for others in the class. In the groups, I’ll also simulate a CGI technique, in a sense that I am trying to elicit their creative thinking with respect to multi-digit multiplication.</td>
<td></td>
</tr>
<tr>
<td>Decision-making</td>
<td>And the questions at the end were just, none of this was really planned. I didn’t write, in my head I didn’t know exactly how it was going to play out. I had to see what they were going to do with the blocks, and then once I realized that with some of those questions kind of what the missing piece was. When I walked into class, several people were discussing/comparing their work. To keep that momentum, I asked you to spend about 20 minutes continuing that work.</td>
<td></td>
</tr>
</tbody>
</table>
and uses as though they are true” (2011, p. 53). Therefore, if someone believes, for example, in the power of education as a tool for social change, then this belief is part of his or her ideology and his or her knowledge base. In education especially, knowledge is often ideologically driven. In my own teaching, I may draw on the work of Paulo Freire, Lisa Delpit, Sonia Nieto, and Gloria Ladson-Billings to talk about the role of race, culture, language, and equity in classroom teaching. The work of these scholars is part of my knowledge base because of the kinds of ideologies I have developed about teaching and learning.

Wilson, Shulman, and Richert (1987) state, “In teaching, the knowledge base is the body of understanding, knowledge, skills, and dispositions that a teacher needs to perform effectively in a given teaching situation” (p. 105). Thus, teachers’ ideological dispositions are a part of their knowledge base, and are therefore included in the category of knowledge.

![Organizational Map of Categories and Subcategories](image)

*Figure 5. Organizational Map of Categories and Subcategories*
Once I placed the category of ideologies into the theme of knowledge, my framework was developed enough to begin capturing the big ideas relating to the goals, knowledge, ideologies, and practice of MTEs. This resulted in a report (stage six) in the form of individual case studies and a cross-case analysis. The framework portrays the complex and collaborative nature of knowledge, goals, and practice. The three major components of goals, knowledge, and practice and the sub components are clarified and described in the next chapter.

Informed by my review of the literature and the analysis of my data, I propose a framework that takes a collaborative approach to examining MTEs knowledge bases through the study of their knowledge, goals, and practice. This approach situates MTEs’ knowledge, goals, and practice as evolving constructs that are reflected on and revised as a result of MTEs’ experiences in and out of the classroom and the complex nature of teaching. The three components of my framework contain subcomponents, as shown in Figure 6 below.

**Knowledge**

For the purpose of describing the knowledge base for MTEs, it is important to move beyond labeling it as conceptual, procedural, or pedagogical. Certainly an MTE needs to have a deep knowledge of the mathematics that he/she is teaching, including theories and historical contexts, but knowledge is not static, it is constructed by the past educational, work, and life experiences of the MTE, as well as everyday interactions with students and fellow colleagues.

Cochran-Smith and Lytle’s (1999) conceptualizations of knowledge make up the next several components of the knowledge theme in my framework. The first conception, knowledge-for-practice, draws up images of knowledge of the subject matter, educational theories, and effective strategies and practices for teaching. In addition, it invokes images of practice in the knowledge-for-practice relationship, which include the way teachers select activities for their lesson, organize the lesson components, and structure the mathematical discourse. The second conception, knowledge-in-practice, describes knowledge as practical and shaped by a teacher’s experiences, purposes, and values. The images of practice are centered on the actions that teachers take amid the changing landscape of a
classroom lesson. These images are different from those related to knowledge-for-practice because whereas the former relies on a fixed knowledge base, the latter is a process of constructing new knowledge in the face of unexpected situations that inevitably come up during teaching.

In addition, new knowledge is constructed during reflection on the actions taken during this lesson that did not go as planned. The last conception, knowledge-of-practice, is knowledge that is constructed collectively during the span of a teacher’s career by studying the practice of oneself and others. This knowledge is often described as transformative because it has the power to shift what teachers believe
about what counts as knowledge, as well as who has all the answers. The images related to practice in the knowledge-of-practice relationship include teachers as curriculum developers, activists, and researchers.

All three of these conceptions of knowledge (of-, in-, and for-practice) are prominent in the knowledge bases of my participants. One of them, Alec, started out as a mathematics major and a teacher of college mathematics and has a strong knowledge base that relates to knowledge-for-practice, in terms of teaching mathematics. But he also tutored all through his school career and later volunteer-taught in sixth grade for two years. These experiences helped him build a knowledge base in the knowledge-in-practice construct. Finally, he is a researcher and a critical thinker. Every day, after class, he writes a summary of the lesson taught, as much for his students as for himself. His teaching philosophy is informed by his knowledge-of and in-practice. While discussing that some universities now have developmental mathematics course specifically for teachers, Alec mentioned that he was initially opposed to them because, informed by his educational and professional experiences, he used to believe everyone should take a certain amount of mathematics in college in order to be college and career-ready. He said his older and more traditional colleagues influenced this ideology of learning mathematics, but as he gained more knowledge from teaching the courses for PSTs, his views toward the path that students should take to be mathematically literate shifted. He explained:

But as I've taught MFET [mathematics for elementary teachers], taken graduate courses, and read, I've realized that though I excelled with traditional math and can actually teach it well that way, at least comfortably, not for the benefit of many students in the class, it isn't what I believe in anymore (post-observation interview, March 9, 2015).

He concluded that he has come to believe that developmental mathematics courses specifically for PSTs are a good idea and can set them up for success in mathematics content courses.

Another participant, Tina went to school to be a high school teacher and taught high school mathematics for many years, thus developing both the knowledge-for-practice and knowledge-in-practice knowledge bases. She develops her knowledge-in-practice by being an active listener and learner. Tina considered students’ thinking as a source of knowledge for herself and other students in the class. She explained:
That’s why I walk around, and I see the different ways of thinking, and I’ll sometimes say, “I didn’t think about it that way. That’s really good. That’s a good way of doing it,” and it extends my thinking too (post-observation interview, May 21, 2015).

Now, as a professor, she engages in lesson study, a form of professional development in which teams of educators work collaboratively to plan and study their lessons in order to improve their instruction and thus student learning. Through working on this research of one’s own and others’ practice, she is developing new knowledge—knowledge-of-practice.

Another subcomponent of knowledge is a category that includes the sources of MTEs’ knowledge. This category includes such sources as MTEs’ own education, field, and work experiences. These sources encompass the practical wisdom component, that Perks and Prestage (2008) identified, but their relationship to MTE knowledge is a complicated one. Tamir (1991) described two types of relationships between personal knowledge and professional knowledge – one having to do with acquisition of professional knowledge and the other having to do with application of professional knowledge. These relationships are an essential part of what makes each MTE’s knowledge base so unique. For example, two MTEs who have similar prior educational or professional experiences may develop different knowledge from these experiences since their interpretations of them are influenced by their preexisting cognitive structures. In turn, these differences are reflected in the application of instructional methods by the two MTEs. Even if the content they choose to teach in a mathematics content course for PSTs is the same (e.g. multiplication of fractions), each MTE may use a different instructional activity to serve a distinct purpose informed by the unique application of his or her professional knowledge.

Conversations with their colleagues, reading academic handbooks and journals, and watching other teacher educators or teachers in action are all sources of knowledge for MTEs. Chauvot (2009) calls this knowledge, knowledge of human resources, and defines it as knowledge of experts in the field whose research connects to the topic the educator is planning to teach. For one of my research participants, Sal, these resources are his colleagues, Diane Hermann and Paul Sally, who wrote *Number, Shape, & Symmetry: An Introduction to Number Theory, Geometry, and Group Theory* (2013), a text which Sal often references when he is planning his lessons. Alec describes the contribution of his predecessor and
office mate, who taught all the mathematics for teachers courses before he retired and Alec took over, to his knowledge of models for operations with rational numbers, “I think my knowledge of the models came a lot from my predecessor’s book because he was very strict about the models, so I have sort of adapted that over the years” (Pre-observation interview, March 18, 2015).

**Ideologies**

In the literature review chapter, I wrote about the importance of attending to the following when teaching and conducting research about teaching and learning: context (where, when, and whom you teach), equity considerations (how well you understand who your students are and how their race, culture, language, and other identity aspects relate to learning mathematics), and disposition towards learning and the learner (how you feel about what, how, where, and whom you teach). I also explained that, except for a few notable examples (Gutierrez, 2013; Martin, 2013), the literature on the preparation of mathematics PSTs does not include these considerations. Because I understand the important role that cultural, race, language identities and dispositions of teachers and students play in educating prospective mathematics teachers, it was my goal to explore the ways in which MTEs address these identities and dispositions in their mathematics content courses. Although the interview questions I developed did not directly ask my research participants about their ideological stances, it was my aim to pay attention to the way that the goals, knowledge, and practice of MTEs revealed their dispositions towards social justice, equity, and identity issues alongside their dispositions about learning and teaching mathematics.

In my research study, I did not directly ask my participants about their ideological beliefs and stances. Instead, ideologies of the participants were inferred since individuals can be unaware of the ideologies that shape their actions. In mathematics education, case study is a popular method to use in order to “infer teachers’ beliefs related to mathematics teaching and learning” (Phillip et al., 2007, p. 450). By relying on a rich dataset of interview, observations, and course documents, I was able to provide a description of my participants’ philosophies and ideologies related to mathematics learning and teaching. The importance of ideologies lies in their being the driving force behind much of the goal-setting, decision-making, and instructional moves. When faced with a challenge, surprise, or other decision-making and invoking
situations, teacher educators’ actions were driven by their ideological stances. Teacher educators’
decisions in response to challenges provided me with some of the evidence from which to infer their
ideological views.

While analyzing the data I noted that my participants’ beliefs about learning and teaching often
formed the backbone of their personal philosophies and approaches to teaching. Factors such as how
educators view the role of authority in knowledge building can influence their goals and practice. For
example, Tina believed in the process of co-constructing knowledge together with her students in order to
make sense of topics and gain deeper understanding. She was engaged in learning with her students, as
opposed to them learning from her while she was learning nothing. Tina understood that learning is much
more productive when it is collaboratively constructed rather than passively transferred. Using this
knowledge, she set teaching and learning goals that resulted in students making sense of the material,
rather than her explaining the material to them. Instead of holding deficit ideologies about her students,
she saw them as valuable contributors to the classroom discourse. This knowledge and goal was reflected
in her practice as she created activity structures where students often presented their explanations of
mathematical concepts to the whole class.

Interviewing and watching my participants teach revealed philosophies, values, and stances, which
illuminated their ideologies about teaching, learning, and mathematics. Alec called it his “agenda” or
“unofficial theme of the course.” Here is what he had to say about why his course focuses on exposing
PSTs to solution methods, which were student-invented and alternative to the standard algorithm:

The unofficial theme of the course is that we want to validate as much as we can the math that
students bring to the classroom. The teacher is still the authority, the knowledge authority at least, but
knowing that students have the abilities to share these things, too. Coming in with that idea rather
than a deficit frame is powerful (post-observation interview, March 25, 2015).

A deficit frame is an ideology that assumes students either know nothing when they come into the
classroom or their knowledge is not as valuable as that of the instructor. The opposite of this frame is the
assumption that students are coming into the classroom with an abundance of resources to offer their
classmates and teacher. Inside this ideology are various beliefs, such as students are capable of knowing
and generating knowledge, students’ knowledge needs to be validated by their teachers, and teachers need to frame their practice in a way that acknowledges and values students’ contributions to the classroom discourse, thus sharing knowledge authority with them.

Before interacting with my research participants and delving into the research literature, I had considered there to be three general types of philosophies of mathematics education. The first belonged to instructors who see mathematics as a beautiful fascinating discipline and, in sharing this view with the students, engage in mathematics work that is not applied in any way. The second type of educators were those who believe that good mathematical skills will help students in college and careers. They apply mathematics to real-world contexts in order to show students its usefulness. The third type of educators believe mathematics is a civil right and should be used as a tool for fighting injustice. They teach mathematics for social justice and encourage students to use mathematics to solve problems in their communities. However, after meeting my participants I was able to see that my view of mathematics philosophies was too narrow. For example, I noticed that some of my participants felt strongly about being a knowledge authority, while others felt strongly about sharing knowledge authority with their students.

The notion of knowledge authority led me to the work of Perry (1970), and in turn to the work of Ernest (1991), who described Perry’s theory of dualism in this way, “Thus in terms of epistemological beliefs, Dualism implies an absolutist view of knowledge divided into truth and falsehood, dependent on authority as the arbiter. Knowledge is not justified rationally, but by reference to authority” (p. 112).

MTEs’ use and development of knowledge is a key component of my research, therefore, I was drawn to Ernest’s (1991) descriptions of dualist, multiplist, and relativist perceptions of knowledge. A dualist sees authorities as keepers of knowledge and knowledge quality, meaning that authority decided whether what one knows is right or wrong. A multiplist believes that authorities can have a certain amount of uncertainty about knowledge and do not have to be all-knowing. Finally, a relativist sees knowledge as something that requires constant evaluation and evidence. Therefore, it is not the responsibility of
authority to state what is right or wrong, rather it is up to the knower to evaluate their knowledge claims and support them with evidence.

Based on these broad conceptions, Ernest (1991) conceptualizes the ideologies into industrial trainer, technological pragmatist, old humanist, progressive educator, and public educator. In describing the goals and practices of my participants, I utilized some of the above-mentioned ideologies, to explain why my participants set certain goals and employed specific practices.

**Goals**

As there is no official program of education that teacher educators must complete or standards for teacher educators to follow. Teacher educators construct goals by making value judgments in regard to the best way to help prospective teachers develop the knowledge needed to teach mathematics (Jansen, Bartell, & Berk, 2009; Hiebert & Grouws, 2007; Hiebert & Morris, 2009). These judgments may be informed by societal expectations; teacher educators’ personal, educational, and professional experiences; the work of researchers; and visions of experts in the field (Jansen, Bartell, & Berk, 2009; Tamir, 1991).

Certainly, different MTEs have diverse ideas of what these goals should be for their PST students. For the authors of The Mathematical Education of Teachers 1 and 2 (2001, 2012), Usiskin (2001) and Ma (1999), the goal of mathematics teacher education is to develop in PSTs a firm, deep, and profound understanding of the mathematics they will teach. For Ball, Thames, and Phelps (2008) the goal is to help PSTs develop mathematical knowledge for teaching, which, in addition to a strong knowledge of the mathematics as a discipline, includes a deep understanding of mathematics-specific pedagogy, as well as knowledge of students. Hiebert, Morris, and Glass (2003) set two learning goals for prospective mathematics teachers: becoming mathematically proficient and learning to teach with growing effectiveness overtime in order to develop mathematical proficiency in their students. Yet, other mathematics teacher educators have still further goals for PSTs. In order to help his students deepen their mathematical understanding, and develop reflective dispositions for teaching mathematics to students in highly diverse urban classrooms, Martin (2013) integrates mathematics content and equity considerations into the mathematics content courses he teaches. Gutierrez (2013) sees developing political knowledge as one of the goals of
pre-service mathematics teacher education thus redefining mathematics and learning as political constructs that affect who learns, what is learned, and how it is learned in the mathematics classroom.

For the purpose of my study, I have divided goals into teaching goals, learning goals, and curricular goals. Within those categories, teaching and learning goals may be mathematical-content specific, related to mathematics knowledge for teaching, or socially oriented. Teaching goals are the goals that the MTE sets for her/himself, such as fostering PSTs’ conceptual understanding of mathematics. Learning goals are the goals that the MTE has for her/his students, such as developing different representations of the solution to the same problem. Mathematical-content goals are content-specific goals that MTEs have for their students or for themselves. Since the content courses I studied were for pre-service teachers, MTEs set goals that went beyond merely knowing and being able to do the mathematics. They included communicating and justifying mathematics procedures and processes and understanding and valuing students’ unique mathematical thinking. Here is Alec talking about a quiz question he created, where he asked the PSTs to give feedback on authentic student work:

The method that the student used on the quiz, adding up, was actually a fairly nice elegant way to do it, so it’s sort of a judgment on your part as the teacher to determine how much you validate. But at the very least validating correct answers and not just saying, “Don’t do it this way,” that’s something that I really want us to think about in this class (classroom observation, March 9, 2015).

Socially-oriented goals are goals related to students’ behavior and dispositions towards mathematics teaching and learning. Goals related to building a community of learners, developing open-minded approaches, and attending to the special needs of students are socially oriented.

Curricular goals are goals related to the curriculum of the course. My participants talked about the fine line between teaching a mathematics content course as either a content or a methods course, since both are aimed at building a knowledge base for teaching. Curricular goals involve considerations about the purpose of mathematics content course, decisions about which topics to present, how deeply should the investigation into each topic go, and which materials should be used to present the various mathematics topics to the students. In addition, curricular goals address the instructors’ selection of texts and handouts. Curricular goals are the reason that Alec and Tina both chose Beckmann’s textbook. They
each stated that they liked her approach to mathematics in terms of operations rather than number systems. Nevertheless, Alec often included models and representations from additional sources in order to expose students to a variety of methods for thinking about a mathematical concept. Sal did not use a textbook at all, deeming all the textbooks for PSTs he had encountered as incomplete in terms of the depth of content knowledge he thought suitable for PSTs to develop.

**Practice**

Practice is an integral part of knowledge. Quoting Schön, “. . . our knowing is ordinarily tacit, implicit in our pattern of action and in our feel for the stuff with which we are dealing. It seems right to say that knowledge is in our action” (quoted in Cochran-Smith & Lytle, 1999, p. 263). Schön (1987) explains that the main goal of reflection is to learn from practice by problem solving on instructional moves taken during teaching. Through the process of framing and reframing certain aspects of their practice MTEs learn and their knowledge evolves. Schön (1987) calls this reflection-in-action and explains that the practitioners have to problem-solve the aspects of their practice they aim to understand deeper or want to change. This component was very prominent in my research participants’ practice. Here is one of my participants describing his reflection, “Despite the fact that I have taught this course so many times, there is still a couple things in it that I debate the order of, and what will work best, and this [multiplication] is a good example of that. All the pieces make sense individually but where do I fit them together in this portion of the course?” (Alec, post-observation interview, March 23, 2015).

The instructional moves and activity structures of the MTEs are closely tied to the constructs of knowledge and goals. Based on the MTE’s beliefs about who is in control of the knowledge in the classroom, he/she will make decisions about who presents the information that is to be learned to the class. MTEs who believe that students can generate knowledge, rather than merely consume it, will ask them to present their solutions to the class, or structure discourse in way that accounts for the knowledge that PSTs bring to the classroom. On the other hand, MTEs who feel that their job is to transfer the knowledge they develop to students in the form of tips and best-practices may choose to present material to the PSTs in a way that does not promote exploration or alternative solution methods. Additionally,
MTEs conceptualization of knowledge construction itself influences their use of instructional moves and activity structures. Teacher educators who believe in constructivist theories of knowledge development will structure their class sessions in a way that allows for students grappling with concepts and discovering mathematical properties and processes. MTEs who draw on sociocultural theories will facilitate activities that allow students to work together to discover different mathematical topics through conjecture, critique, and analysis of each other’s theories. Finally, MTEs who believe students have no role in knowledge construction use direct instruction to pass knowledge along to their students.

The last component of the practice theme is MTEs’ decision-making. Westerman (1991) describes decision-making related to one lesson as consisting of three stages. According to Westerman, decision-making takes place while planning the lesson (pre-active stage), while delivering the lesson (interactive stage), and while reflecting on the lesson taught (post-active/how successfully stage). The quality of decisions made in each stage is essential to the lesson’s success because “the quality of people’s decision making—in problem solving, teaching, and most everything they do—affects how successfully people attain the goals they set for themselves” (Schoenfeld, 2011, p.36). Therefore, decisions are directly tied to goals at every stage. Here’s Alec talking about the in-the-moment decision he made to veer off his plans, “The bigger challenge was definitely the fraction division, but they went with me down that road. But I had to revise, because I wasn't planning for it to take so long. I thought that they would get it a little bit quicker. That's why I kept throwing more stuff at them. I try to push them, and I realized, no, they still, it was so far from their reality, that they need, I felt like they [needed that]” (post-observation interview, April 22, 2015).

By analyzing the practices of instruction moves, activity structures, and decision-making of the four MTEs in my study, I learned how each practice is related to their goals, knowledge, and ideologies. In the case study chapters, I illustrate these relationships between practices, goals, and knowledge, by displaying segments of their lessons in a table format. This shows connections between particular interactions, such as between an MTE and his or her students, and the goals of the MTE, as well as the role of knowledge and ideologies for particular classroom vignettes.
The more we can learn about MTEs the better we can understand how to improve the profession. Although I conducted a cross-case analysis of the participants’ data, my goal was not to generalize but to describe the practice and knowledge of individual MTEs who teach math content courses to elementary PSTs through the lens of the framework described. The three components of my framework are interrelated and work together so that, if there is a shift in one or more components, there is a shift in the others. These shifts may impact one of the components or both simultaneously. For example, something that takes place during a class session can generate new knowledge and prompt for goals to be revised. On the other hand, development of new knowledge through an experience inside or outside the classroom may inspire new goals, which in turn will influence the instructional moves of the MTE.

Saxe (2014) explains that individuals shape and reshape their goals as practices take form, then generate goals that contrast in character as a function of the knowledge that they bring to practices. This view, of goals as “emergent phenomena, shifting and taking new form as individuals use their knowledge and skills” (Saxe, 2014, p. 17), together with a conception of knowledge as developing and evolving and practice as something that teacher educators work to improve by constantly reshaping their goals as a result of their growing knowledge, provides a lens for study of the knowledge base of mathematics teacher educators.
CHAPTER 5

GOALS-KNOWLEDGE-PRACTICE: CROSS-CASE ANALYSIS

In this chapter I will discuss the findings related to the components of the framework—goals, knowledge, and practice—for each participant and relate these findings to each other. I begin the chapter by describing the findings about the knowledge the four MTEs in my study draw on in their teaching of content courses for pre-service elementary education students, and explain how these forms of knowledge are related to their goals and practices. Next, I describe each MTE’s teaching, learning, and curricular goals and highlight the particular knowledge forms and ideologies that inform these goals. Finally, I explain how the MTEs’ practices of instructional moves, activity structures, and decision-making are related to their goals and knowledge.

A diverse set of participants allowed me to give rich examples of the work of MTEs using the goals-knowledge-practice approach. I would like to recall some of the key characteristics of the four MTEs in my study, in Table IV, below.

Analysis of the data for all for participants revealed the complexity of the work of teacher educating and MTEs’ efforts to navigate this complexity given the lack of a shared knowledge base for teacher educators. The common cross-analysis themes include several shared goals, such as deepening the PSTs’ mathematical content knowledge and building PSTs’ confidence in their mathematical abilities. When it came to MTEs’ practice, the theme of the struggle between the content and pedagogy sides of courses for teachers was evident for all participants. A related theme was the participants’ solution to this challenge, which was to model instructional strategies while teaching the content as a way of inserting pedagogy into the course. Finally, in terms of the knowledge, each MTE shared their uneasiness about not having a formal education for teacher educating. They each talked about learning the content but not necessarily how to develop this content knowledge in others, or as Sal put it “I learned a heck of a lot of math but not necessarily about the teaching of math” (initial interview, January 28, 2015).
To describe how they developed knowledge for teacher educating, each talked about learning from educators they looked up to and imitating their goals or instruction styles. Each of the participants also talked about evolving ideologies, refining their goals, and deepening their knowledge, in order to continuously improve their practice. The participants suggested numerous differences between the knowledge needed for teaching mathematics and the knowledge needed for teacher educating (such as an understanding of how adults learn and the ability to revisit old ideas in new ways). Realizing these differences, teacher educators worked hard to gain the latter knowledge. The acknowledgement of the differences between the two knowledge bases and the teacher educators’ desire to attain the knowledge.
needed for teacher educating point to a need for formal professional development for teacher educators. By situating my work within the research literature, I am able to describe how this knowledge is related to the goals and practice of the MTEs in my study.

It is across the examination of the four cases that the complexity of the relationships between goals, knowledge, ideologies, and practice and their contribution to the development of MTEs’ learning connected to the ideas in the research literature. It became clear that goals are not simply goals, they are a function of knowledge, and knowledge is much more complex than what participants know, it is a function of deeply held ideologies. Upon looking across all the cases and at each one individually, it became apparent that the relationship between goals, knowledge, and practice is not linear. Instead, the components of my framework are collaborative—a change in one or more components usually propels a shift in the others.

Through this multiple-case study, I intend to explain the instructional moves, activity structures, and decision-making done by the MTEs in my study by showing how each constructs their knowledge, goals, and ideologies and applies it to his or her practice.

**Knowledge**

To understand my participants’ goals and practices, I had to understand the wide span of their knowledge. It was important to understand not only what they know but also how they came to know it and why the knowledge is valuable. To understand my participants’ knowledge in this broad sense, I drew on Cochran-Smith and Lytle’s (1999) conceptualization of knowledge and practice, which includes knowledge-for-practice, knowledge-in-practice, and knowledge-of-practice.

**Knowledge-for-practice**

The first conception, knowledge-for-practice, draws up images of knowledge of the subject matter, educational theories, and effective strategies and practices for teaching. In addition, it invokes images of practice in the knowledge-for-practice relationship, which include the way teachers select activities for lessons, organize lesson components, and structure mathematical discourse (Cochran-Smith and Lytle, 1999). This knowledge is gained through formal education, coaching, and mentoring.
All four of my participants hold undergraduate degrees in mathematics and pride themselves on having strong mathematical content knowledge. For example, Fay felt that her mathematical content knowledge was strong enough to petition the mathematics department to allow her to teach mathematics content courses while being a faculty member in the department of education. She felt that because her undergraduate degree in mathematics was earned from the department that she was petitioning. The faculty there, many of whom taught her as an undergraduate, trusted her to teach mathematics well. Alec felt that his strong content knowledge allowed him to criticize and restructure the curriculum in a school where he worked as a new teacher. In describing the required mathematics curriculum of the elementary
school where he volunteer-taught as oppressive, Alec declared, “I thought, “I am a math expert, and I think this is nonsense”” (initial interview, March 6, 2015). When discussing how the mathematics teachers and mathematics teacher educators have the hardest jobs, Tina elaborated, “It’s hard for me too . . . Sometimes it makes sense, and sometimes it doesn’t make sense. I have multiple ways of checking it out. If all else fails, if I can’t think about it logically, I go back to procedural. I can do procedural” (post-observation interview, June 2, 2015). So even though she is not able to always understand how a student approached the problem, she has enough confidence in her mathematics abilities to know that she can, at least, check the answer in multiple ways. Sal’s expertise is evident through his confidence to use his own judgment, rather than a textbook or guidelines, to decide on the content of the course. When I asked him how he decided what the PSTs in his course would learn, he answered,

What, in my professional opinion, is that a student should know about the basic concepts of geometry, like angle, area, classification of shapes, triangles, congruence, similarity, right? Those are somehow the essential there in six words, the essentials of plane geometry that it seems like any teacher of any level of mathematics should know (initial interview, January 28, 2015).

Although, their content knowledge was strong, most of the MTEs in my study felt that their education did not prepare them to teach teachers mathematics to aspiring teachers. Sal explained that he was fortunate enough to have a teacher educator as an instructor of the course that taught him how to be a university lecturer. He felt that a vast amount of his instructional approaches to educating teachers came from observing the teacher educators whose work he admired. He titled his mentors and role models as “people who do great stuff in teacher education” (initial interview, January 28, 2015).

Similarly, as a new teacher educator Alec learned a lot from another educator in his department. When I asked Alec about his extensive repertoire of models and representations for mathematical operations, he explained:

I have to thank my prior officemate, getting into the nitty-gritty of representing fractions and the arithmetic, I could have picked it up myself with the text, but having somebody there modeling it for me . . . I never watched him teach it, but we would have conversations in the office about it and play with the snap cubes and do that, so he was kind of teaching me how to teach that (final interview, May 14, 2015).
Tina cited her education as a source of some of her knowledge, like the knowledge of some of the theories that she applies to her practice. She explained that, “In the doctorate program, it was theories of learning that we had talked about. So Bruner came up, Piaget” (post-observation interview, May 21, 2015). Tina regularly sought out professional development opportunities in order to expand her knowledge base. During one of our post-observation interviews, she exclaimed with excitement “Is it 3 o’clock yet? I am going to take a webinar, NCTM, on multiplying fractions with manipulatives!” (Post-observation interview, May 5, 2015). Another time, she told me about a class she was taking to learn more about online learning. She explained that it was important for her to keep her knowledge base current and learn about how to take advantage of modern technology to enhance her courses. In fact, her teaching goals often included the use of technology:

I have a YouTube video around division, and it is designed to be kind of funny but to make people think about different interpretations . . . I want to use the eBook with the Smartboard. My teaching goal for today is better use of technology” (pre-observation interview, May 5, 2015).

Other participants referred to learning from conferences, professional development sessions, talking to their colleagues, and their own reading of academic literature. For example, Fay regularly attends and presents at various conferences. She explained that she felt she learned “a lot from talking to other, obviously, faculty that are . . . more veteran teachers, have been doing it for a while” (initial interview, January 27, 2015) and often applied something new she learned at a conference to her practice.

Both Sal and Alec described themselves as self-learners. Alec added “I like to just kind of sit down and figure things out, but if I've got a community of people that are deliberately meeting with homework, I'll do it and I'll learn from it, I'll really be thankful that I did” (final interview, May 14, 2015). In fact, several of the participants said they like to figure things own on their own. Tina also looked up topics as they came up, adding, “You get that knowledge base when you need it” (post-observation interview, May 21, 2015). The semester that I observed her, she taught herself how to use the software called GeoGebra because she felt that her students would benefit from using it. The way she learned about GeoGebra is through her knowledge of human resources (Chauvot, 2009). Before becoming a teacher educator, Tina was the director of a national grassroots-based mathematics intervention project for several years.
Because of this experience, she has worked with a number of different mathematicians, and one of them taught her to use the geometry software. This mathematician is someone she can call on to help her build a stronger knowledge base around geometry, as she explained below:

There is a book that, this summer, I think I’m going to study, and it’s around geometry, and it’s a really nice book. It does it conceptually, and then my friend in Cornell, he’s retired now, but he does something on spherical, on spheres, that kind of thing. I think in the fall we’re going to do something with his work. He’s working with teachers this summer, but in the fall we’re going to have to take over that work. He is phenomenal (post-observation interview, June 2, 2015).

Knowledge of human resources is another common thread that all of my participants shared. Chauvot (2009) defined knowledge of human resources as knowledge of “experts in the field whose scholarly work reflected specific topics” (p. 368). Each MTE in my study was fortunate to know people whose expertise they can utilize when teaching mathematics to pre-service teachers. For Tina, these people include David Henderson, Deborah Ball, Sybilla Beckmann, and many others. For Alec, these are former and current professors, his former officemate, and his wife, who is a schoolteacher. As a graduate research assistant, Fay worked closely with Barbara and Robert Reys and learned about developing students’ number sense, a goal that is still prevalent in her teaching. Sal cites his colleagues, Diane Hermann and Paul Sally, as his most valuable resources. In planning his lessons, Sal often references the textbook Hermann and Sally co-wrote, Number, Shape, & Symmetry: An Introduction to Number Theory, Geometry, and Group Theory (2013). Nevertheless, although my participants came to teacher educating with an enormous amount of knowledge-for-practice, many of them explained that they draw on their practical knowledge for planning, delivery, and reflection.

**Knowledge-in-practice**

Lacking official preparation for becoming a teacher educator did not stop my research participants from becoming effective practitioners, but it did make it more likely that they learned from their practice, as opposed to their teachers. The second conception, knowledge-in-practice, describes knowledge as practical and shaped by a teacher’s experiences, purposes, and values. The images of practice are centered on the actions that teachers take amid the changing landscape of a classroom lesson. These images are different from those related to knowledge-for-practice because, whereas the former relies on a fixed
knowledge base, the latter is a process of constructing new knowledge in the face of unexpected situations that inevitably come up during teaching. In addition, new knowledge is constructed during reflection on the actions taken during lessons that do not go as planned (Cochran-Smith and Lytle, 1999).

In terms of teaching experience, prior to becoming teacher educators, all of the MTEs in my study taught college-level mathematics courses to non-education majors. Alec has also taught elementary school mathematics for two years, and Fay is currently getting her secondary certification and student teaching in eighth grade. Sal has taught endorsement courses to in-service teachers, but has no experience teaching at the school level. Tina taught high school mathematics for twenty years and college mathematics for one year before she became a teacher educator. Fay, Alec, and Sal still sometimes teach mathematics courses in addition to teaching mathematics-for-teachers courses.

All of my research participants agreed that teaching mathematics to PSTs requires preparation that they did not receive formally. Tina remembered keeping one chapter ahead of her students when she first started teaching courses in the department of education. However, she quickly realized that she was familiar with much of the material that she was teaching. For example, she was already using the classroom management technique of standing close to students who were being disruptive to get them back on track, but she did not know the name for that technique (proximity). Through experience teaching the education courses, she learned what the dean that hired her knew already—being a high school teacher had equipped her with a segment of the knowledge base for teacher educating. She told me that all she had to do was connect the two knowledge bases, which she did by naming the methods she already knew how to implement, observing that,

In most cases, I was already doing that, but I didn’t know what to call it. It was like, okay, so you have a situation where you make sure that this person is ready to learn and so, I didn’t know that it was a name for that. I would say, “Oh okay.” Or in terms of giving them simpler but similar examples. I intuitively knew to do that. My brain didn’t work like that because I didn’t think that way, but then when I got there I was like, “Oh, so Polya says, “give them a simpler problem.” That makes perfect sense.” I’ve been doing this for a while (post-observation interview, May 21, 2015).

For Tina, the knowledge was already there, but the goals evolved from engaging students to teaching PSTs how to engage students. This shift in goals created the need for additional knowledge, in this case in
the form of naming techniques she was already using. In her practice, in addition to using the techniques,
she also had to be explicit about her actions. This skill of making tacit knowledge explicit requires a high
level of metacognition on the part of the teacher educator (Smith, 2003).

From teaching elementary school students and college students Alec learned that the approach to
teaching the two audiences about numbers has to vary because of each group’s unique foundational
knowledge. He observed that,

The trouble with teaching college is that students are coming in with fragmented knowledge already.
This is very different. That's why some of the ed. theory is tough to apply to the college teaching
setting because this isn't the first time, second time, third time they're seeing this material. They've
seen it hundreds of times potentially, and they've developed all kinds of habits, really unbelievable
habits whereas the 6th or 7th grader that's seeing laws of exponents the first time, it could very well
be their actual first time doing it. It's easier almost to negotiate the why with that audience than it is
with the college student who just wants to get a good grade in the course if they're really extrinsically
motivated (initial interview, March 6, 2015).

By saying this, Alec confirmed that the knowledge for teacher educating is not the same as the
knowledge for teaching at the school level. Deep knowledge of mathematics, alone, will not be enough to
understand and repair the PSTs’ fragmented knowledge that Alec was referring to. Thus for him, a shift in
awareness brought on a shift in knowledge and practice. In his syllabus, he writes:

All of you have a very different mathematics history and, as such, “mathematical baggage.” If we
don’t address some of this baggage here, there’s a chance that you’ll carry it over to your own
classroom, which could have terrible consequences (Mathematics for Elementary Teachers I syllabus,
winter 2015).

Once he understood that that college students come with “baggage,” Alec knew he had to gain the
skills to bring that baggage to the surface and the knowledge to be able to help them gain understanding
of old ideas in new ways. He started using pre-assessments to gain an understanding of students’
knowledge and experience with mathematical topics. In addition, he deliberately sought out tasks that
could be used to show PSTs that there are multiple ways of approaching problems and multiple solutions
paths. Finally, Alec developed a repertoire of representations and models and encouraged students to be
open-minded and engage with unfamiliar-to-them representations.

Out of all my participants, prior to becoming instructors of mathematics courses for teachers, Sal had
the least formal education related to teacher educating and almost no experience teaching teachers. He
was inspired and guided by the teacher educators around him, but his greatest teacher was his own experience teaching mathematics courses. He elaborated:

I haven’t read Ball. I know basically what it's all about from conversations, but um, I'll quote an old another mentor of mine, when I was at Harvard. He said, “Teaching mathematics is in some ways more difficult than knowing mathematics or doing mathematics because if you are a mathematician, you just have to know how to do something right. If you are a teacher, you have to understand all the wrong answers too. You don’t just have to know the right answer, you have to know all the different ways that students can mess something up or misunderstand it” (initial interview, January 28, 2015).

As stated in Cochran-Smith and Lytle (1999), many aspects of his knowledge are tacit, “implicit in action and artistry—that artistry itself is a kind of knowing” (p. 263). Sal identifies with the artistry aspect of knowing so much so that he compares certain aspects of his knowledge of his craft to that of the work of an actor in a staged performance:

Part of what makes an actor a good actor is the emotional sensitivity to what everybody else on stage is thinking. It's not just what they're doing, but it’s the reaction to their fellow actors and to, frankly, sometimes to the energy that’s coming from the audience too depending on the size of the building and the theater. There’s different kinds of silences in a math classroom, right, and you can tell. That’s the silence of we’re just kind of bored. That’s the silence of, “Oh my god, we have no idea what’s happening” (final interview, March 18, 2015).

Sal’s education equipped him with extensive mathematical content knowledge, but it was not until he started teaching that his friend’s words rang true: knowing mathematics is not enough for teaching it. Because of his conceptual education, Sal was able to present topics in multiple ways and link them to each other to help PSTs gain an understanding of how mathematical topics connect. In his mathematical content courses for teachers, he demands that students not merely know the Pythagorean theorem, but understand every aspect of it, and can prove it in multiple ways:

What I want you to understand is that the Pythagorean theorem is a theorem about areas and specifically that it’s the area of these squares. It was known simply as Ancient Chinese proof because it was found on a manuscript. And at the bottom there was a single Chinese symbol, which means Eureka! It is a proof of Pythagorean theorem. It’s now up to you to do 2 things: one, modern algebra is a very useful way of proving this, and two, more important is the geometric argument, cutting out the shapes (lesson observation, February 17, 2015).

Thus, what they lacked in formal knowledge, my participants made up for with practical knowledge, defined by Fenstermacher (1994) as "what teachers know as a result of their experience as teachers"(p.1), which is distinct from the knowledge gained from research that has been produced by others for them to
use. In the case of Sal, that knowledge came from experiences teaching mathematics to non-education majors as much as it did from knowledge gained working with in-service teachers, as shown in his description of these experiences, below:

I taught a course in undergraduate calculus in the number of variations. I did a specialty course for the undergraduates in number theory and geometry that was beyond the kind of courses that we are talking about now for the teacher program. Being able to create coursework, I think that’s probably the most significant thing, is really getting the independence to be a standalone lecturer from doing those kinds of things. So it had everything to do with my formation as a teacher in that sense: the freedom to experiment, write problem sets, and think about different ways of teaching things, different examples, different technologies, and so on . . . so I had an opportunity to get in and work in a program like SESAME and work with these teachers, and it just completely changed my thinking. I mean, that was a radical change: to realize the kinds of knowledge that these teachers, these in-service middle-grade teachers were bringing to the table and the kind of needs that they had in terms of both content and methods of delivery, and so on. So that, just, that was probably the biggest change that happened during graduate school—my opportunity to be a TA with Paul the teacher in 1997 (initial interview, January 28, 2015).

Alec confided that prior to teaching teachers he had never thought about why the base 10 number system works in the way it does, why fractions get inverted when divided, or what is the purpose of common denominators. He also did not give much thought to the different representations that one problem can offer. When I asked Alec about his transition from pure mathematics to teacher education, he explained that he learned much of what he teaches the PSTs while teaching mathematics for teaching courses, adding,

As I was teaching, I would discover all these things and get really excited about it. Think about all the stuff that I do now that I've learned in these last eight years about this content. It's remarkable. It's a whole body of knowledge that I didn't have before (initial interview, March 6, 2015).

Tina credited her students with being a source of her learning, as she described here, when I asked how she encourages students who are struggling to gain skills and confidence in their abilities in mathematics:

You see I have these boards and I’m working a problem and it’s like, “Oh, oh, yeah, so let me see how you did it.” They’ll show me and I’ll show them what I did, and it’s like co-learning. Although I should know this stuff and I . . . . But it’s like, okay, so she’s learning too or she’s going to engage in this with me. And so that’s the thing to kind of engage in it. And sometimes I’ll engage, and I’ll say, “You know this is the first time I thought about it this way. This is really nice to do it that way.” This is the way I typically do it but this is really . . . . It’s not a co-learning but it’s a co-experience type situation where I’m participating with them. Even if it’s 3 of them, I’m participating (post-observation interview, June 2, 2015).
Tina has known for some time that good teaching calls for accurate listening—the skill of listening to and understand and appreciate the thinking, feeling, and actions of others and appropriately respond in an effective manner. Through this practice, and her goal of developing multiple representations, Tina continuously gains new insights and grows her knowledge base. These shifts in knowledge bring about shifts in goals and practice. As she learns from her students she adjusts her goals and instructional moves to meet their collective needs.

Finally, each of the MTEs used reflection as a source of constructing new knowledge, implementing new goals, and improving their practice. Alec described himself as a reflective educator, and explained that as he became more confident in his teaching his reflection became more encompassing, adding,

It's just a matter of knowing what to do with that critical reflection has changed over the years whereas at the beginning, it may even be like, "Oh, I'm going to change this technique, that technique." Now it's like, "I'm going to change the philosophy of what I'm doing in this course (initial interview, March 6, 2015).

When Alec first joined his current mathematics department, the required text was written by his officemate, and although Alec learned a lot about modeling mathematics from his colleague, he felt that the text as a whole was “antiquated” and “not appropriate” (post-observation interview, March 18, 2015). Alec explained that it read more like a College Algebra text and not a text for PSTs. He added, “So I learned a lot from him, but I also learned about how to work with this audience, a potentially math-anxious group” (post-observation interview, March 18, 2015). At this point in his career Alec’s view of himself was shifting from someone who teaches others mathematics to someone who teaches others how to teach mathematics. Additionally, his teaching ideology was shifting from the old humanistic view of mathematics as pure and void of attention to the individual learner to a progressive educator view of mathematics as a discipline that can encompass the creativity, diversity, and past experiences of his students (Ernest, 1991). Therefore, his goals and his practice had to catch up to his shifting knowledge and ideology. In describing this shift, he proclaimed:

I think what I was trying to say is that there are two views of math "living" in me. On the one hand, I grew up on traditional math instruction and skill-based mathematics with algebra as "god." However, in my teaching of dev. ed. courses and my experiences with students, I've moved away from the traditional view of "math for math's sake" and algebra being all about manipulation, mechanics, and
drill/skill. Now I try to contextualize, push the conceptual, and entice them into the subject through inquiry (post-observation interview, March 9, 2015).

He started by making a curricular decision to switch textbooks, and explained that this instructional decision allowed him to change the course philosophy so it aligned with the more current philosophy of the author of the new text. In his practice, Alec stayed true to his promises of exposing students to the mathematics they will teach in a new way by ridding them of the baggage they bring to the classroom, which included outdated beliefs about mathematics teaching and learning, a lack of confidence in their skills fueled by anxiety, and an education that was more procedural than conceptual. In fact, one of his overarching goals is to challenge students’ perceptions of mathematics and mathematics doers. The instructional moves to reach this goal include starting class by asking PSTs to solve a problem any way they like as opposed to teaching them a procedure, having PSTs examine videos of students solving problems in ways alternate to the standard algorithm, and having them read and react to articles about teaching mathematics. Evidence of this shift in his practice can be found in a writing assignment prompt that asks students to read and reflect on an article about teaching the distributive property:

Comment on the following sentence at the end of the article. “Unfortunately, we often jump from one method to the next without making connections between the methods or to the underlying ideas that hold them all together.” In particular, relate your past math experiences with the sentence. Did your past experiences as a student resemble this? (writing assignment 8, April 6, 2015).

Because Alec is a reflective educator, he asked his students to reflect as well, as shown with the above question where he asked them to reflect on their past experiences with mathematics in order to connect these experiences to teaching mathematics to their future students. During a different class session, he asked students to write a short reflection remembering how they learned multiplication, then used their reflections in the next lesson to talk about going beyond teaching the facts in terms of teaching multiplication. This practice of constant reflection helped Alec develop new knowledge and gave him ideas for strategies to use in future class sessions. However, the biggest source of new knowledge developed came from work than involved knowledge-for-practice.
Knowledge-of-practice

Knowledge-of-practice is knowledge that is constructed during the span of a teacher’s career by studying the practice of oneself and others. This knowledge is often described as transformative because it has the power to shift what teachers believe about what counts as knowledge, as well as who controls the knowledge. The images related to practice in the knowledge-of-practice relationship include teachers as curriculum developers, activists, and researchers (Cochran-Smith and Lytle, 1999).

Of the four participants, Tina and Fay are currently engaged in research. Alec plans to perform research, and Sal is not engaged in research of his own, preferring to concentrate on teaching instead. In the future Alec plans to research teacher education courses for pre-service teachers, but for now, his knowledge-for-practice comes from his participation in a mathematics education study group with MTEs from a range of higher education institutions. The focus of the group was developing video-based learning environments for teacher education courses. Alec appreciated this opportunity to co-construct knowledge with other teacher educators. He said, “It wasn't like I was being given this knowledge; having conversations, I was actually getting a chance to network and talk to other math teacher educators, which I think was probably the most valuable of all of it” (final interview, May 14, 2015).

Fay’s deep understanding of developing conceptual knowledge and mathematical knowledge for teaching come from her own research. While teaching developmental mathematics at a community college, she studied, wrote about, and presented on aspects of her practice that pertained to developing her students’ conceptual understanding of mathematics. Currently, she is studying the practice of teachers, from the time they take methods courses as PSTs to their current teaching assignments in various schools, in order to learn about the ways in which teachers gain and use mathematics knowledge for teaching throughout their career.

In her own practice and in the behavior of her students, Tina, is always aware of the competencies, that she believes make a good teacher: conceptual adaptability, active empathy, and cultural understanding. While working as the director of a national grassroots-based mathematics intervention
project, she was part of the team who conducted the research that resulted in identification of these competencies for teachers and trainers. She describes the competencies in the following way:

Those are the competencies when you say, so what makes a really good math teacher? Then another concept would be around whether or not people are active listeners, and so that became the accurate empathy. Then another one would be around whether or not you were able to think about the culture of people you were in, and sometimes challenge those norms because for us in math, it’s challenging norms around, “I’m not good at being a math person. I’m not doing good.” Sometimes you have to challenge it. Those are the three big broad areas that we’re in (post-observation interview, May 21, 2015).

Her involvement with the research around competencies resulted in an ideology shift that, in turn, impacted her goals and practice. Whereas before she was drawing on a variety of experiences and literature in order to shape the PSTs in her classes into effective practitioners, she now had a particular knowledge base to draw from, in order to structure her goals. Her goals became more focused and efficient. The new goals included developing PSTs’ abilities to be conceptually adaptive, be an accurate listener, and have cultural understanding. In her practice, she designed activities to reach these goals and pointed out to the class when someone exemplified a quality well by telling them when they were being conceptually adaptable or an especially good listener. These activities and goals were tied to her deeply-held beliefs that all students can learn mathematics and how best to teach mathematics. Tina’s beliefs and the beliefs of the other MTEs in my study, about mathematics as a discipline and mathematics teaching and learning, led me to consider ideologies as a component of knowledge.

Ideologies

Because teachers’ beliefs and philosophies about the nature of knowledge and knowing, also called ideologies, are often the backbone of their personal philosophies or approaches to teaching, I have chosen to include ideologies as a subcategory of knowledge. Factors, such as how educators view the role of authority in knowledge-building, can influence their personal epistemologies and practice. For example, Tina believed in the process of co-constructing knowledge together with her students, in order to make sense of topics and gain understanding. She was engaged in learning with her students, as opposed to them learning from her while she is learning nothing. Tina knew that learning is much more productive when it is collaboratively constructed rather than passively transferred. Using this knowledge, she set
teaching goals for herself that incorporated students making sense of the material, rather than her explaining the material to them. This knowledge and goal was reflected in her practice, as she created activity structures where students often present their explanations of mathematical concepts to the whole class. Similarly, Fay felt that her students’ knowledge is valuable enough to be voiced alongside her own. She told them so on the first day of class by stating, “Your ideas are important. And so if you have an idea that's different from something that you see me do or say, I want you to be able to share it” (post-observation interview, February 5, 2015). This ideology was reflected in her practice, as I often observed students speaking up to share methods that were different from hers and asking questions about the validity of her solution and explanation.

Some research in the field describes personal epistemologies in terms of context-specific epistemological resources (Louca, Elby, Hammer, & Kagey, 2004). Through this view, it is believed that educators can have manifold ways of knowing, depending on the learning context. The teacher educators in my study have highly sophisticated personal epistemologies due to their wide-ranging experiences with learning and teaching. This is important, since studies show that personal epistemologies are consistent with teaching practices (Brownlee, 2004; Kang, 2008; Mius, 2004). Therefore, a mathematics teacher educator who believes there to be only one way to solve mathematical problems, that knowledge is facts, and that the teacher is the ultimate authority will exhibit teaching practices which value memorization of rules and algorithms, rather than conceptual understanding and modeling, and use lecture over exploration as an activity structure choice. On the other hand, an MTE who believes problems can be approached in a variety of ways that all lead to the same correct answer, mathematics is a sense-making discipline, and students have a role in the construction of knowledge, will structure his or her class to allow students to explore and discover, with a concentration on understanding, justifying, and modeling mathematics, rather than memorizing it.

Ernest (1991) proposed the following ideologies as a way of understanding educators’ philosophies of teaching and learning mathematics: industrial trainer, technological pragmatism, old humanitarian,
progressive educator, and public educator. In the sections that follow, I highlight only the ideologies that I found my participants to have developed, based on the analysis of the data.

**Old humanist ideology**

In the separated relativistic absolutism of the old humanist ideology the emphasis is on proof as a way to show that mathematics is a body of absolute truth. This is often called pure mathematics and conjures images of incomprehensible symbols written on a blackboard that only a handful of old white men with crazy hair can decipher. It positions mathematics as elite and only for the select few and ignores issues of race, culture, and class as they pertain to learning and teaching of mathematics. This ideology is best represented by Sal’s approach to teacher educating.

In his responses to my interview questions and his teaching of the course, I regularly observed Sal pointing out the beauty in mathematics and the general public’s lack of understanding of the truth that is geometry. In describing the essential knowledge needed by future elementary teachers in the area of geometry, Sal proclaimed:

> There is a beauty, well, so there is a structure to things too, so part of the structure of geometry is the axioms, the definitions leading to the axioms, leading to the theorems, and I think that builds appreciation for the way mathematics gets done and the way new mathematics gets created and the way that mathematics is interrelated to itself. So it's not even about geometry, per se, but making them have an appreciation for the truth of mathematical statements, the nature of what it means to say that this formula is true, what it means to have hypotheses. You know, I’d say if you ask the population at large: what does the Pythagorean theorem say? Ok, there's going to be a certain segment who won’t know, fair enough. But I would say that 90 plus, may be 99 percent of everybody else who actually do know will say, “$A^2 + B^2 = C^2.$” And my response to that is, “Oh I thought it was a theorem about triangles. Where is the triangle in your equation?” (initial interview, January 28, 2015).

Sal believes that mathematics should be understood, not for some means or end, but for its own sake. Before the final class session, I asked Sal about the knowledge he draws upon in order to help students reach the goals he sets for them. He replied, “All of geometry, from the ancients (Egyptian, Babylonian, Greek, Chinese) to the modern (Gauss, Bolyai, Lobachevsky, Hilbert)” (pre-observation interview, March 3, 2015). His response of citing the widely accepted, and, in some sense, famous mathematics civilizations and mathematicians shows an inclination towards the old humanist ideology, which values knowledge as set by widely recognized authorities.
Alec started out in the field of pure mathematics but his ideological beliefs about mathematical knowledge has shifted as a result of his work as a teacher educator and several professional development opportunities he engaged in about mathematics knowledge for teaching. His course emphasized student-invented algorithms as a way to show PSTs that mathematics is not cut-and-dry, or black-and-white, but rather is influenced by the individual nature of the students learning it. By exposing the PSTs to a variety of alternative algorithms, he was emphasizing the role of identity, race, culture, and class in the mathematics classroom. He urged his students to pay attention and praise the efforts of students whose solution methods are unique and original.

**Progressive educator ideology**

The connected relativistic absolutism, or the progressive educator, ideology of mathematics is a progressive position—knowledge is viewed as absolute, but this perspective takes into account the individual’s role in understanding and knowing the subject. This is often represented in education as a “child-centered” perspective and values creativity, diversity, individual expression, and personal knowledge. With this ideology in mind, educators seek out tasks and problems that propel students’ interests in a discipline. For example, wanting to know how to predict events, such as population growth or decline, should motivate students to study patterns and functions. At the core of this ideology is the “call for teachers to do more listening as they elicit student thinking and assess their understanding and for students to do more asking and explaining as they investigate authentic problems and share their solutions” (Feiman-Nemser, 2001, p. 1015). Alec’s determination to bring students’ questions, ideas, and solutions to the forefront of his practice exemplifies this ideology. He explains that he often throws his plans to the wind to follow a student’s idea or inquiry and that this action makes him the kind of educator he is today.

After reflecting on the semester taught, Fay has decided to put more emphasis on students teaching each other. She told me that she plans to have students make presentations of new concepts to their peers. This is a major shift in her practice, since normally Fay introduces every new concept and then gives the
PSTs time to practice it individually or in groups. When I asked her what influenced this new goal of student presentations she replied:

Obviously, I don't want to take away from the class being content-focused, but I feel if they have to explain something to their peers or create an activity around explaining or introducing a concept, I think they'll remember it better and understand it better than me explaining it (final interview, May 27, 2015).

She added that she is looking forward to students forming a deeper connection to the content they will be assigned to explain. In addition, by designing the lesson segment activities and delivery, students will be able to practice their pedagogical skills, Fay added:

To me, it's not only that they know the content, obviously, but I wanted them to be able to explain it to someone else, and that's going to reinforce their own content knowledge. But it's also going to tap into, obviously, pedagogical knowledge because they're learning pedagogy at the same time, a little bit of it. I mean, just enough to give them a flavor (final interview, May 27, 2015).

Alec exemplified this ideology by asking students to solve problems any way they knew how, instead of requiring that they use one method. When students came up with methods that were different from his own, he praised their method and asked them to elaborate on aspects that were new and original. He also urged his students to appreciate and value students’ unique ideas by having them examine student work and give feedback that showed they want to learn more about the non-traditional method. After handing back a quiz that asked such a question, Alec reminded the PSTs of their responsibility to their students by saying,

Be careful with the types of things you are saying to a student who is using an alternative strategy. That student was correct, but some of you, in the tone of your writing, the student may have taken that as feeling bad about his method. Our job is to validate good mathematics if we can (classroom observation, March 9, 2015).

In his initial interview, Alec stated that although he has read much about the topic, he wanted to do more around teaching for social justice. I think that by stressing the PSTs’ responsibilities to validate students’ unique solution methods, often influenced by their background experience, he was moving his practice in the direction of his aspiring goal.
Public educator ideology

Relativistic fallibilism, or the public educator, is a critical ideology that sees knowledge as socially constructed and mathematics as something that can be improved upon and used for a greater good. Educators who have developed this ideology often use mathematics as a tool for social justice and equality. This ideology takes mathematics out of its “ivory tower” and puts it into the hands of any man, woman, or child, from any racial, cultural, or linguistic background.

There are many approaches to being a public educator. Banks (2004) describes citizenship education by speaking to the importance of diversity and unity in the development of students’ knowledge, attitude, and skills needed to function in a rapidly changing society. Citizenship education, he states, should help students remain attached to their cultural, racial, ethnic, and linguistic identity, but also allow them to effectively participate in the shared culture. The balance between diversity and unity is a delicate one, Banks explains:

Unity without diversity results in cultural repression and hegemony. Diversity without unity leads to Balkanization and the fracturing of the nation-state” (p. 291). A public educator aims to foster practices that promote democratic education in his or her classroom. Distinguishing qualities of democratic classrooms to include: (a) problem solving curriculum, (b) inclusivity and rights, (c) equal participation in decisions, and (d) equal encouragement for success (Beyer, 1996; Pearl and Knight, 1999; Wilbur, 1998).

By incorporating equity, social justice, and civil rights into their teaching, public educators promote learning that is just and transformative and that encourages their students to think critically and apply what they have learned to the classroom to help their community.

In terms of equity considerations in mathematics content courses, some of my participants stated that they could be doing more but are restricted by a lack of knowledge, resources, and time.

Alec, critical of himself in response to my questions about the theories that drive his instruction, stated:

I also believe much of what I’ve read recently with respect to teaching and learning in K–12. That’s not to say that I’ve been able to apply all of it. For instance, I believe in social justice education but have been unable to truly incorporate that in the MFET courses other than through critical analysis of reforms and curricula. I could likely push the envelope a bit more, but there’s only so much time (initial interview, March 3, 2015).
For Alec, the knowledge of social justice and equity is there, but he lacks time and resources for implementing this knowledge into practice. This is not atypical. Issues of culture, race, language, and identity are perceived by MTEs to be addressed in their education courses. Nevertheless, although Alec may not use the words *social justice* in his teaching, he openly identifies changing students’ perceptions about mathematics and mathematics doers as a goal for his course. When I asked him to explain what that means to him, he answered by stating some of the characteristics I listed earlier when I described the public educator ideology—he made mathematics accessible to his students and wanted to shift their beliefs from mathematics as something that is mystical and not meant for them to understand to a discipline that they take pride in knowing and are confident in teaching to their students. He added:

I try to help them see that anyone with the desire can learn math. For them, this takes the form of knowing that they can transition from 1) someone who rushes through a problem hoping that their answer is correct (and that correctness must be verified by the teacher), to 2) someone who takes time solving problems and investigating his/her thinking, to 3) someone who feels like (s)he can create the problems, thereby taking control of the mathematics. The other half of this is challenging students' perceptions of who does math. Is math something done just by old white guys or by anyone? Most K–8 teachers are women. I don't think many of my students hold the perception of women being worse than men with respect to STEM, but that false perception still exists. I want them to be able to easily dismiss that. The same goes for perceptions along socioeconomic lines (post-observation interview, April 6, 2015).

Informed by her work as the director of a national grassroots-based mathematics intervention project, Tina explained that she used “the social justice and urban educators’ models to demonstrate how to work with those who don’t routinely get quality mathematics experiences and education” (initial interview, March 23, 2015).

In response to my inquiries about the theories and perspectives that inform her teaching, Tina responded:

I use the social justice and urban educators’ models to demonstrate how to work with those who don’t routinely get quality mathematics experiences and education. This is done in a variety of ways including: encouraging high quality math teaching to all students, teaching to students' strengths as opposed to weaknesses, and teaching using quality curricular materials (Initial interview, March 23, 2015).

One of these models, a multi-step learning process, is adapted from her work as a director of a national grassroots-based mathematics intervention project. First, students have an authentic experience
that relates to the concept they are studying then students draw a picture of the experience. Next students write about the experience, first in colloquial language, called *people talk*, and later in academic language, referred to as *feature talk*. Finally, the students express their thinking in symbolic language, thus making the transition from mental math to algebraic thinking. The purpose of the multi-step learning process is to demystify mathematics and help students see that they have access to the same information and knowledge as everyone else if they learn to decode it.

To help students understand the concept of average, Tina applied the multi-step learning process to take students from physically stacking and leveling out cubes to solving average problems mathematically. In the process she had them write about what they did by asking what they think the purpose of the activity was (*people talk*), and then writing about the process of stacking and leveling off and its connection to finding the average (*feature talk*). During the next class, she uses the following prompts to remind the students about the physical and mathematical experiences:

**Warm up problem:**
- Demonstrate the “leveling off” process to find the mean for the following data that represents the number of apples in 5 baskets: 2,3,4,5,6.
- Use a diagram with stacks of unifix cubes.
- Use numbers only. (Lesson slide, April 9, 2015).

She then had students reflect on the purpose of using each method and engaged in metacognitive behavior by sharing her knowledge of a specific learning process, revealing her own understanding of why it’s important to start with the concrete and move to the abstract.

Other participants stated that they have never considered making conscious decisions to address citizenship or democratic education in a mathematics classroom, which is common, especially if they learned how to be a teacher educator by watching others, who likewise did not make these considerations.

I would like to close this section by pointing out that the relationship between mathematics teacher educators and ideologies is not a one-to-one function. Each of the participants in my study can be linked to multiple ideologies. For example, most of the data collected for Sal’s case portrays his inclination toward the old humanist ideology, this includes his goal of studying mathematics for its own sake and his knowledge base as consisting of mostly traditional and widely-accepted methods and authorities.
However, during several interviews, Sal expressed a desire to have students think about mathematics from a variety of different standpoints and consider a range of methods that can be used to prove theorems. During a lesson on Pythagorean theorem, he presented the PSTs with several different proofs and encouraged them to take different approaches to starting the proof. Alec aligned most closely with the progressive educator ideology, but many of his goals and practices can also be associated with the public educator ideology. All this is to say that one MTE can engage with numerous ideologies at the same time, and ideologies can shift and evolve throughout an MTE’s career.

In addition, there are some ideologies that are not described by Ernest (1991). For example, he does not explicitly talk about teaching as service in the way that Alec speaks about his work. When I asked him to explain his decision to volunteer-teach, he explained that after 9/11 he felt a calling to volunteer, and after considering his skill set, decided to seek out an organization through which he could volunteer-teach mathematics. In describing the alternative certification program, he chose, Alec said:

It was a faith-based community—spirituality, simple living, and teaching-as-service—thinking of teaching and service at that point, which was good. Of course, there's a fine line between teaching-as-service and the idea of being a missionary like Martin talks about. I had some of those things to grapple with in my first years of being in School of Ed. Was I going to this like a missionary? Part of my failings probably were because of that mindset, and it's hard not to have that mindset when you first start out even though if you're well read in radical philosophy and in reading Delpit and whoever else. It's hard to resist that temptation (initial interview, March 6, 2015).

Alec, is aware that his disposition toward teaching-as-service may come off as a deficit ideology of savior and is even accepting that he may have held that view toward his students in his volunteer-teaching experience. Now, as a teacher educator, he is very careful to cultivate the appreciation and understanding of students’ understandings, previous experiences, and alternative algorithms, without framing these aspects in a deficit way. As his knowledge-in-practice developed, so did his goals and ideologies. He no longer wishes to be a missionary, but rather he wants students to realize that their knowledge, no matter how different from the status quo, matters in his classroom.

Additionally, he wants PSTs to realize that their knowledge impacts their students. This is how he addresses PSTs who feel that they already know mathematics, and are resistant to learning mathematics in the deep way he wants them to learn it. While discussing fixed versus growth mindset, Alec shared with
me that he often has to explain to students just how important it is to know mathematics for teaching. He makes this point at the expense of sharing his own failure.

They go, "I know this stuff. How could I not be able teach it?" Then, I have to tell my own story to say, "I majored in math until the 6th grade, and I failed miserably during those 2 years." I really feel like, I can count the good lessons on two hands. I'm probably being overly critical, but seeing what I perceive as good teaching now, I know that that's not what I was doing. And part of it is, it was my first two years teaching period, but still, it was two years of teaching. It was a year in a student's life that I influenced. That's a hard thing (initial interview, March 6, 2015).

Goals

Since there are no official programs of study or standards for teacher educators to follow, they must construct goals by making value judgments concerning the best way to help prospective teachers develop the knowledge needed to teach mathematics. These judgments may be informed by societal expectations; teacher educators’ personal, educational, and professional experiences; the work of researchers; and visions of experts in the field (Jansen et al, 2009; Tamir, 1991). Certainly, there exist diverse ideas of what these goals should be for PSTs. For the authors of The Mathematical Education of Teachers I and II (2001, 2012), Usiskin (2001), and Ma (1999) the goal of mathematics teacher education is to develop in PSTs a firm, deep, and profound understanding of the mathematics they will teach. For Ball, Thames, and Phelps (2008) the goal is to help PSTs develop mathematical knowledge for teaching, which, in addition to a strong knowledge of the mathematical content, includes a deep understanding of mathematics-specific pedagogy, as well as knowledge of students. Hiebert, Morris, and Glass (2003) set two learning goals for prospective mathematics teachers: becoming mathematically proficient and learning to learn to teach with growing effectiveness overtime in order to develop mathematical proficiency in their students. Yet, other MTEs have still further goals for PSTs. Martin (2013) integrates mathematics content and equity considerations into the mathematics content courses he teaches with the goal of PSTs increasing their mathematical understanding and developing reflective dispositions for teaching mathematics to students in highly diverse urban classrooms. Gutierrez (2013) sees developing political knowledge as one of the goals of pre-service mathematics teacher education, thus redefining mathematics and learning as
political constructs that effect who learns, what is to be learned, and how it is to be learned in the mathematics classroom.

In my study, I examined MTEs teaching, learning, and curricular goals. Within those categories, teaching and learning goals may be mathematical content specific, related to mathematics knowledge for teaching, or socially-oriented. Teaching goals are the goals that the MTE sets for her/himself, such as asking students for different representations of the solution to the same problem. Learning goals are the goals that the MTE has for her/his students. Mathematical content goals are content-specific goals that MTEs have for their students or for themselves. Curricular goals are goals related to the curriculum of the course. Alec set the following course goals of for his students in a mathematics content course for pre-service teachers: “I want them to know the inner workings of addition, subtraction, multiplication, division, and pretty much any rational number within reason—knowing the nuts-and-bolts, ins-and-outs of pretty much all of the arithmetic an elementary student would see” (initial interview, March 6, 2015).

In addition to describing what he wants his students to be able to do, Alec constructs his teaching goals in terms of his students’ learning goals. His syllabus states:

It is my aim to help you . . .
1. Select, apply, and translate among mathematical representations to solve problems.
2. Communicate orally and in writing their mathematical thinking coherently and clearly to peers, teachers, and others.
3. Apply and adapt a variety of appropriate strategies to solve problems, including technology.
4. Make and investigate mathematical conjectures individually and collaboratively.
5. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole (Mathematics for Elementary Teachers I syllabus, Spring 2015).

As will be shown in detail in his case study, Alec’s practice is closely connected to his goals. Many of the activities his students do in class require them to work with and understand different mathematical representations, apply a range of strategies, investigate claims and make conjectures about problem outcomes, and connect mathematical ideas to each other. Alec engages in professional development to align his knowledge to his goals. He attends workshops on multiple representations, joins special interest groups that focus on the education of pre-service mathematics teachers, and does his own research around how other instructors teach mathematics content courses for prospective teachers.
Alec’s lesson plans always included a mix of mathematical goals related to content knowledge and knowledge for teaching. One of the first things he did when he came to class was write these goals on the board so that his students know what the class session outcomes will be for each lesson. For example, during a week focused on multiplication, Alec’s goal’s for the first class session of the week were: “Justify the standard alg. for multiplication; Examine student errors with the multiplication algorithm; Define powers/exponentiation; Model fraction multiplication” (Lesson Plan, April 6, 2015). The goals of justifying, defining, and modeling fall under developing specialized content knowledge for teaching (Ball et al., 2008), but one could argue that a good mathematics instructor may want to develop these skills in all students, not just prospective teachers. The goal of examining student errors, however, falls under content of knowledge and students (Ball et al., 2008) and is clearly aimed at helping students develop mathematical knowledge for teaching. Because of his ideological dispositions toward mathematical access for all, many of Alec’s goals centered on understanding and valuing students’ unique mathematical thinking. In talking about a quiz question he created, where he asked the pre-service teachers to give feedback on authentic student work, Alec explained:

The method that the student used on the quiz, adding up, was actually a fairly nice elegant way to do it. So it’s sort of a judgment on your part, as the teacher, to determine how much you validate. But at the very least validating correct answers and not just saying, “Don’t do it this way.” That’s something that I really want us to think about in this class (observation, March 9, 2015).

This quiz question illustrates a number of goals Alec had for the PSTs: goals related to actually knowing how to do the mathematics in order to judge the accuracy of a method; goals related to exposing students to alternative methods to show them the individual nature of student learning; and goals related to pedagogical knowledge, such as giving constructive and productive feedback to students, in order to correct or advance their thinking.

Tina consistently attended to both mathematics and mathematics-for-teaching goals in her class. Before a lesson on proportional reasoning, I asked her why she was ending the course with this topic. She explained that students’ had a very procedural understanding of ratios, proportions, and percent, and so it was important to revisit these ideas in a conceptual way. Additionally, she explained that understanding
of proportional reasoning connected many of the ideas in elementary mathematics to each other, and one of her teaching goals was to make the PSTs aware of the multiple connections of mathematical ideas across different topics. She told me that students would resist learning this material because it would not be on the final exam, but that she would tell them:

> As teachers that’s what you need to know. Part of what we do is build up to the progressions, especially algebraic thinking, and that progression is mostly rates of change. And so when you think about rates of change you can think about them in different ways—they could be proportional, they could be inverse, they could be a number of things, directly proportional where they are both increasing, both decreasing, one can increase, the other can decrease. And so these last two sections kind of lends itself to that thinking (pre-observation interview, June 4, 2015).

When I asked Sal about his teaching and learning goals, he stated the content goals for his students. His focus on deep mathematical content knowledge was clear from the initial interview. When I asked him about the goals he has for the students in his course, Sal stated:

> I want them to deepen their knowledge of geometry. I want them to walk away knowing more geometry than they did when they came in. Or at least be reminded of a lot more, so yeah, so deepening maybe even in the revisiting sense. Maybe they didn’t actually learn all that much more content wise but they've thought about it differently. So maybe this is answer two, actually, just flat out more knowledge. I hope they know more geometry than when they came in (initial interview, January 28, 2015).

Many of the goals Sal set for his students were connected to the old humanist ideology, which stresses mathematics as a pure body of absolute truth. They were centered on proof, history of mathematics, and deep knowledge of foundations, such as constructions and precise definitions. However, the quote above alludes to his desire for students think about mathematics differently and aligns more with the progressive educator ideology, which values different individual perspectives, creativity, and diversity.

The above example showed how Sal’s seemingly content-related goals can be interpreted to illuminate his focus on mathematics teaching, which includes encouraging students to be creative in theory solution paths. The goals for a lesson on Pythagorean theorem, however, clearly show that Sal thought about content knowledge and mathematical knowledge for teaching when planning the segment, as stated below:
The goals are to teach them about the Pythagorean theorem. Not that I don’t doubt that they already know it but to really get them to understand that it is a theorem about area, not a theorem about numbers. And then also to show how we can use certain basic tools in the classroom to prove geometry theorems (pre-observation interview, February 17, 2015).

During this session, his practice showed a focus on mathematics-for-teaching goals, as most of the lesson session was dedicated to the PSTs using construction paper, glue, and scissors, to experience proof through the eyes of their prospective students. Although not stated explicitly during the pre-observation interview, his other goal was to show PSTs a variety of approaches for proving the Pythagorean theorem. In doing this, he took something that was seemingly familiar and traditional and made PSTs see and experience it in new ways. During the class session, he engages PSTs with geometric, algebraic, and visual proofs. He concluded the lesson by explaining how what they did in class will tie to the homework:

Two problems for next week. One is find another proof of the Pythagorean theorem. In addition to its importance, it’s also famous because of how many proofs. Two, take this exercise that we just did with the construction paper, ruler, scissors, and write it into a lesson plan. I mean there to be a lot of flexibility. Use this idea in some way to teach this to your students (classroom observation, February 17, 2015).

After showing students that there are a variety of ways that Pythagorean theorem can be proven, Sal asked the PSTs to add to their understanding of the theorem by finding one more way. In addition, he wanted them to think about the different ways their potential students can engage with the proof by asking them to design a learning experience based on their work in class.

Whereas Sal’s goals centered on PSTs seeing the beauty, structure, and multiple ways of looking at mathematical concepts, Fay’s goals revolve around number sense and deep conceptual understanding. In college, her work as a research assistant, under mathematicians who were studying mental math and number sense, exposed her to research on the importance of number sense and the different strategies that help students develop it. She later adapted this work to her teaching of remedial mathematics courses to college students, writing papers and presenting at conferences to share her findings with the mathematics community. When I asked her about her goals for her students in the mathematics content course for pre-service elementary teachers, she replied that, influenced by her experiences with number sense, it is a goal for her to develop a strong sense of number sense and answer reasonability in PSTs.
Additionally, Fay stressed the importance of conceptual understanding and often stated her goals for
the individual class sessions in a way that highlighted the attainment of conceptual understanding. For
example, in response to my routine question about her teaching and learning goals during our pre-
observation interviews, Fay replied:

I want students to understand the multiple meanings a fraction can have (part of a set, division, part of
a whole, distance on a number line, ratio). I think it is important for my students to be able to teach
the many representations of a fraction. This is important because most students do not understand all
these representations, therefore, struggle with a conceptual understanding of fractions (pre-
observation interview, March 24, 2015).

Her practice often supported these goals by asking conceptual questions. Before every exam, Fay held
in-class review sessions, during which she handed students study guides with conceptual questions, such
as the ones below:

1. Explain why the method of scratch addition works.
2. How can base 10 blocks be used to teach addition with carrying and subtraction with borrowing?
3. Explain why the lattice multiplication algorithm works.
4. How is short division similar to long division? How is it different?
5. Explain why the repeated subtraction algorithm works for division (Math for teachers study
guide, Chapter 3).

During these class sessions students worked in groups to answer and present certain questions to their
peers. Students’ discussing mathematics was an important teaching goal for Fay, and students’
understanding of different perspectives, methods, and solution paths was a learning goal achieved through
students’ discussions of the conceptual questions.

Although each participant went about it in a unique way, their course goals focused around several
central objectives. However, no matter how it is phrased (Fay and Alec called this conceptual knowledge,
Tina called it conceptual adaptability, and Sal referred to it as essential knowledge), each of them is
determined to develop a deep conceptual understanding of mathematics in the PSTs they teach. To
achieve this level of understanding each MTE stressed the importance of explanation and multiple
representations. Their goals included having PSTs justify, model, prove, and make connections in order to
gain a deeper understanding of methods and procedures. The word justify appears 31 times in Alec’s
lesson plans of 32 class sessions over 16 weeks. For Sal, the objective for every lesson was to prove a
mathematical concept. Most class sessions included students completing geometric, algebraic, and visual proofs in order to promote deeper knowledge through multiple representations. Tina often referred to students’ sense making through multiple representations. Many of the times I observed, although her lesson slides instructed students to represent an operation in several specific ways (such as table and equation) she would ask the students to represent in all the other ways they have learned (such as scenario, double number line, graph, or annotated equation). Fay’s focus on conceptual knowledge and number sense showed that her goals included students understanding how to do the mathematics procedures, but also why the process works and how to make sure the answer suits the question. These are some of the ways that all of my participants worked to develop conceptual knowledge using unique methods informed by their knowledge base, experiences, and ideologies. Another overarching goal that I observed each of the MTEs in my study strive to achieve was a socially-oriented goal that had to do with dispositions towards mathematics as a discipline and mathematics learning and teaching.

Socially-oriented goals can be predetermined or emergent, overarching or local (Schoenfeld, 2000) and are related to students’ dispositions and behaviors, as well as the mathematical content. For example, my participants’ socially-oriented goals included Alec’s goal of challenging students’ perspectives of mathematics and mathematics doers, and Tina’s goal of developing students’ abilities to rely on themselves and each other to determine if something is correct. These goals were predetermined and overarching. In the section of his syllabus devoted to students’ expectations, Alec stated, “Do everything possible to succeed. More than this, keep an open mind to your ability to succeed and believe that you can” (Math for Elementary Teachers 1 syllabus, Spring 2015). Due to the overwhelming persistence of mathematics anxiety in our society, it is not surprising that MTEs focus on developing PSTs’ attitudes, as well as mathematical content knowledge. The desire to develop confidence in their abilities and comfort with communicating using mathematics was a shared theme for my participants.

Although each of the research participants alluded to wanting to develop positive dispositions toward mathematics in PSTs, this goal of building confidence with mathematics and pedagogical abilities was front-and-center for Tina. During our first interview, when I asked her to tell me about the goals she set
for the PSTs in the course, she replied, “I think the first goal is to get people at some kind of comfort level. It's not only comfort level with themselves but comfort level talking to each other about problems, getting things critiqued, because that's really hard sometimes” (initial interview, March 23, 2015). Her other goals included deep mathematical content knowledge and an understanding of school mathematics and how topics connect to one another. To achieve these goals, she drew on her knowledge of teacher competencies. She explained:

The qualities or competencies that a good mathematics teacher educator (MTE) should possess are accurate listening, cultural understanding, and conceptual adaptability. The MTE has to be able to understand the thinking, feeling, and actions of their students in order to appropriately address, engage, and react to their instructional needs. The MTE should also know something about who the participants are and know about their mathematics stories. This knowledge of students’ culture will provide the MTE opportunities to understand the cultural norms and appropriately challenge those norms, if necessary. Finally, the MTE should be conceptually or intellectually flexible. This means that the MTE should be able to provide several ways to represent and interpret mathematically concepts and demonstrate that skill fluently (initial interview, March 23, 2015).

For Tina, PSTs’ comfort level with doing and talking mathematics was the number one goal of the course. Tina is acutely aware of her students’ feelings towards mathematics. Because of this, on the first day of class she initiates a conversation about their attitudes towards mathematics. Many of her students tell her that they are bad at math or that they hate it. Then, throughout the semester, Tina makes sure to draw attention to these students’ productive mathematical ideas and explanations. Her language, when she speaks to the PSTs, also reflects this goal of gaining more confidence. When she sees that several students have trouble using a picture and an equation as representations that that are connected, she tells them, “What I would do is go back and try to work on some of those until you feel more comfortable” (classroom observation, May 5, 2015). The competencies, accurate listening, cultural understanding, and conceptual adaptability, are a major part of building PSTs’ confidence in their ability to do and teach mathematics. By listening to PSTs and valuing their stories and varied methods, as well as expecting them to develop these competencies in their own teaching, Tina is engaging her students in confident building activities and practices. During one of our conversations, Tina explained how she plans to help students think more contextually. When I asked he why that was important to her, she replied:
It is important because that flexibility is something that teachers need. You need to be able to say, “If I have 17 divided by 5 then that means that I have 17 objects and I am actually going to put those 17 objects into 3 different groups, so 17 people into 3 different groups and how many people would I need?” And so in that case instead of the 3, you'd have a 4, and you should answer the question according to the context. I think it kind of relates back to literacy, so a lot of people say that I am not good at math, it's really a reading problem. That’s a basic skill that teachers need to work on (pre-observation interview, May 21, 2015).

The goal of developing mathematics literacy skills and battling mathematics anxiety was somehow embedded into every lesson Tina taught.

Every time, I asked about course or lesson goals, Sal responded with some form of wanting students to know more mathematics, and in a deeper way, but there were quite a few instances when a conversation that was not goals-focused revealed his goals in regard to students’ attitudes toward mathematics. In one instance, Sal referred this attitude as a hope rather than a goal, in another he said, “So it sounds silly, but I want folks to like math. I think that this course is an opportunity for them to find something to like about mathematics even if they didn’t before” (initial interview, January 28, 2015).

Overall, he certainly thought deeply about PSTs’ enthusiasm and engagement levels when it came to the mathematical content he presented. During one of our conversations he said that when a teacher does not like mathematics it shows. When I watched Sal teach, I could tell clearly that he had a passionate, deep love for mathematics, and this translated into his instructional moves and decision-making. When students volunteered a method or a solution path, Sal often asked them to illustrate it on the board, eager to see how their ideas compared to his. When students came into the class discussing mathematics tutoring sessions, which were a part of their degree work, Sal often made the decision to engage the entire class in a discussion of how to best approach a topic that the student was struggling with. Sal had a rough outline for each class session, but his enthusiasm for mathematics and teaching, and his deep content knowledge, allowed him to swiftly change directions and topics.

This brings me to my last section on curricular goals. In one interview, Sal explicitly addressed the shift in goals, which stemmed from his experiences teaching courses for mathematics PSTs:

I guess even in the 5 years now that I have been teaching it, I feel like my, the evolution has definitely come from: “you guys need to know this much geometry,” to much more of nowadays it's: “Look, I want you to not be afraid of math, I want you to appreciate math. I want you to be able to do a certain
amount and our main topic is geometry but really I want you to be walking into the classrooms as confident happy math teachers” (post-observation interview; February 24, 2015).

Alec had a two-pronged approach to the idea of comfort. On the one hand, he promoted the idea of growth mindset and told his students that anyone who has a strong desire to learn math can do so. He had a process for moving their mindset forward. He began by encouraging PSTs to use familiar-to-them methods and models to explain operations, even if these procedures were not traditional. Then he had students justify and model operations to further deepen their understanding and level of confidence. Finally, Alec had PSTs create their own problems as a way to show them that they are capable of not only doing mathematics but also writing mathematic problems for others to do.

On the other hand, he often asked them to get out of their comfort zone and approach problems they had never encountered before in ways that were less comfortable. During one of the class sessions, he had them study Mayan number systems in order to make them understand how their students must feel when they first encounter the base 10 system. During another class session, he asked students which method they felt most comfortable using when performing a certain operation, then chose a method they did not name to challenge them to get out of their comfort zone and engage in a method that was not easy for them.

Fay used her experiences failing mathematics in high school as inspiration for her goal to make sure that students had positive dispositions toward mathematics. She told me that on the first day of class they play a game called “Two Truths and a Lie.” Everyone in the class writes down two statements that are true and one that is not true. One of her truths is always the fact that she failed mathematics in high school. She told me,

They never pick that out as being true. They think that’s the lie. And so that starts a story about how you may think you are confident at some point in your life in something, whatever it may be. Someone questions that. It takes a good person to bring you back to try and explore it again, and maybe in a different way (post-observation interview, February 5, 2015).

Fay believes that by sharing this story she makes herself, as a mathematics instructor, and mathematics itself more approachable. When I asked her to tell me about her course goals, she stated that she wanted to get students excited about mathematics, then added that in addition to having a larger and
stronger content knowledge base, she wants her students to leave the course with a comfort level they
may not have had before. She shared another story with me, on this topic:

You know, I have one student who religiously comes to me at least once a week before class to get
one-on-one help. And when he first started coming to me, he would physically shake. Math scares
him that bad. But now he has gotten much more comfortable. To me that’s what it’s about—creating
that environment where they feel ok with getting help and talking to each other and not feeling
embarrassed about it (post-observation interview, February 5, 2015).

Thus by paying attention during observations and asking the right questions during interviews, I
learned that my participants prioritize improving PSTs’ attitudes towards mathematics and their self-
confidence in their abilities as doers of mathematics, just as highly as they do mathematical content
knowledge.

The last component of the goals category is curricular goals. Curricular goals are goals in regards to
the curriculum of the course. Sal did not use a required textbook for the course but instead utilized the
works of Euclid, Hilbert, and other great mathematicians as bases for his lesson plans. When I asked why
he did not use one of the mathematics for elementary teachers texts, he replied that he has not found one
that he liked. When I asked him to elaborate he explained:

It's not interesting enough. It's not deep enough. And it's not that this text or the manipulatives, or the
handouts or anything like that are incomplete in any sense. I don’t mean that, but that’s not what I am
going for. It's not a course in Euclidian geometry that you might have taken in 10th grade. That’s not
what we are trying to achieve here. We are trying to talk about geometry in all of its aspects and we
will revisit some important topics from the 10th grade, Euclidean geometry class certainly, but that’s
not what it's about (Initial interview January 28, 2015).

A lot of his curricular choices were based on his deep knowledge of mathematics and the connections
between topics and areas of mathematics. When I asked him about the knowledge he drew on to make
planning decisions about the topics he was going to teach, he mostly quoted areas of mathematics, such as
group theory or abstract algebra. Sometimes he would explain that he took some of the teaching strategies
of the mathematics educators he admired and adapted them for his own use. To that point, Sal sometimes
used the book *Number, Shape, and Symmetry* (2013), written by two mathematics educators from his
institution who have served as mentors and role models to him.
Fay used *A Problem Solving Approach to Mathematics for Elementary School Teachers* (2013), as the text for the course. She explained that she chose it because of how conceptual it is and because it has an accompanying workbook that she can use in the mathematics content and methods courses. Because problem solving is in the title of the text, I asked Fay what role problem solving plays in the course. She replied:

The first section of the textbook is strictly called problem solving, but that doesn't mean problem solving doesn't exist throughout. I like that this author starts with problem solving, because right, initially, from the start, we put the focus on the idea of problem solving being important. When I revisit them again in math methods class, they hear about how important problem solving is again (initial interview, January 27, 2015).

Problem-solving and conceptual understanding were the two main foci of the course I observed Fay teach. Because conceptual understanding has been a passion of hers for a long time, many of the materials she used to develop conceptual understanding in PSTs was developed by her over time and for different courses, including the remedial mathematics courses she taught at a community college. This material included conceptual question study guides. Additionally, because of Fay’s wide range of experiences teaching mathematics—at a community college, at a fine arts college, as a coach of in-service mathematics teachers—she has accumulated a wealth of curricular resources she used when applicable.

Tina and Alec both used Sybilla Beckmann’s *Mathematics for Elementary Teachers with Activities* (2014) for their courses, but their decisions to use this text came from different places. When Alec first came to his current institution, he was told to use the text that the other mathematics instructor in the department had written. Alec tried to use it but felt that it was outdated and not student-friendly, so he supplemented most of the time. When the author of this text retired, Alec was told he could choose any text he wanted. He chose the Beckmann (2014) text because some of the MTEs he admired at other institutions used it. Nevertheless, because he has already developed a curriculum made out of a variety of supplemental sources, he continued to use many of them, as he explained, “IMAP folks or the CGI folks, so Franke and Carpenter. Lampert, the usual . . . .” (initial interview, March 6, 2015). Additionally, because he is constantly evolving his knowledge base by seeking out new materials and resources, he is
continually adding supplemental material to his course as he finds new material he thinks will engage his students and develop their mathematical knowledge.

Tina did not choose the Beckmann (2014) text, but instead was told she had to use it by the mathematics department faculty. Because the mathematics content courses for teachers are taught in the mathematics department, Tina had to defer to this department about curricular decisions. Tina told me that at first she did not like the book because she was not used to its unique structure based on operations rather than number systems. However, after using it for a few semesters, she grew to appreciate its organization. In comparing it to another text she used for other mathematics education courses, she said:

I think the way she structured the book . . . . The other book, it was more work with different number systems. This one is more around operations. Yeah. It makes sense. It makes sense to me, especially with progressions. It makes sense to do it this way, and logically, because she's a mathematician, she's going to think that way (initial interview, March 23, 2015).

Tina had previously used some of the representations in the textbook, such as the pattern work around understanding operations with integers, in her work with a national grassroots-based mathematics intervention project. She told me she was glad to see that Beckmann used the patterns as well. In other cases, she felt that some of the activities could be enhanced and has added components to them. On the first day of class, Tina had her students do a stacking and leveling off activity to gain a conceptual understanding of the meaning of average. When I asked her if this activity came from the textbook, she replied that it did not, but she has spoken to Beckmann about adding it. For Tina, the activity of stacking with a fixed amount of cubes and a fixed amount of time provides PSTs with multiple ways of approaching the concept of average, which is necessary for being an effective teacher.

During my observations of the content courses, my participants talked about the fine line between teaching a mathematics content course as a content course and a methods course since both are aimed at building a knowledge base for teaching. Curricular goals involve considerations about the purpose of a mathematics content course, decisions about which topics to present, the depth of investigation into each topic, and choice of materials to be used to present various mathematical topics. In talking about teaching division to her PSTs’ prospective students, Tina asked the PSTs in her class if they are familiar with
children’s literature on the topic: “Have any of you read the book The Doorbell Rang? [It] is a good book that can start a discussion about division. Have any of you taken Children's Literature?” (Observation, May 5, 2015). In this lesson, Tina chooses to go outside the course textbook with the goal of exposing students to curricular materials they can use in their own classroom and connecting her course to the students’ other teacher education courses. This shows how her knowledge of curriculum influences her goal to expose PSTs to teaching methods, as well as develop their mathematical knowledge. This goal manifests itself in her practice by integrating literature into mathematics lessons.

**Practice**

Practice is integrally related to knowledge. Quoting Schön, “Our knowing is ordinarily tacit, implicit in our pattern of action and in our feel for the stuff with which we are dealing. It seems right to say that knowledge is in our action” (quoted in Cochran-Smith & Lytle, 1999, p. 263). Schön (1987) explains that the main goal of reflection is to learn from practice by problem-solving on instructional moves taken during teaching. Through the process of framing and reframing certain aspects of their practice, teachers learn and their knowledge evolves. Schön (1987) calls this reflection-in-action and explains that the practitioners have to problem-solve the aspects of their practice they aim to understand deeper or want to change. When I asked if he would make changes to the lesson he taught, Alec told me, “Despite the fact that I have taught this course so many times there is still a couple things in it that I debate the order of, and what will work best, and this [multiplication] is a good example of that. All the pieces make sense individually but where to fit them together in this portion of the course?” (Alec, post-observation interview, March 23, 2015).

To address the practice component of my framework, I will first describe the subcomponents of practice: activity structures, instructional moves, and decision-making. Following this description, I will illustrate how each component is informed by the goals and knowledge of each MTE.

Instructional moves, of the MTEs in my study, were related to their knowledge, goals, and ideologies. For example, Alec had knowledge of the Cognitively Guided Instruction (CGI) work, and was deeply influenced by their principles, which include encouraging students to use any tools they wish in a way
that makes sense to them, asking students to solve a problem any way they can, and eliciting students’ creativity (Carpenter et al, 1999). His knowledge of this work was clear in his goals and practice. Some of the time Alec clearly communicated the connections between his goals, practice, and knowledge of CGI materials. When stating the goals for a lesson on whole number multiplication, Alec said, “My goals are to provide a space for the students to examine the multiple ways to multiply multi-digit numbers . . . In the groups, I’ll also simulate a CGI technique in the sense that I’m trying to elicit their creative thinking with respect to multi-digit multiplication (pre-observation interview, March 25, 2015). During some of the lessons, Alec told students which materials he was using and why. During a lesson on fraction division, Alec displayed a slide showing multiple solutions to the same problem and told the PSTs, “This comes from this book, Cognitively Guided Instruction, CGI. The whole point of this instruction is that teachers give students tasks that are often beyond their grade level and give students the space to solve them” (classroom observation, April 27, 2015). In explaining this to the PSTs he was also explaining some of his own instructional moves, which involve giving students difficult tasks and providing them with space and manipulatives to investigate the problems on their own.

These are examples of when Alec made the connections of his goals and practice to CGI clear, but there were a number of times when he employed the techniques and strategies used by the developers of CGI without communicating the connections to me or his students. Before introducing a new topic, such as operations with integers or proportional reasoning, Alec would write several problems on the board and ask students to solve them in any way they could or wanted to, thus eliciting their creativity. Other times he brought different types of manipulatives to class (such as blocks, fraction strips, and unifix cubes) and told students that he is specifically not telling them which manipulatives to use because he wants them to use any tools they feel are most useful to them for modeling fraction subtraction and addition. His lesson goals often ask students to explore, compare, and examine different methods in addition to the standard algorithms. Alec told me that reading the CGI texts added to his knowledge base of alternative strategies and encouraged him to make it a point to explore these strategies with his students.
Fay’s goals and instructional moves are influenced by her knowledge of the work on number sense, which she engaged with deeply, while working as a research assistant for prominent researchers in the field of number sense. Because of this knowledge she is able to choose examples that emphasized mental mathematics in addition to the algorithm that she included in students’ notes for chapter three, as shown in Figure 8, below.

![Mental Mathematics: Multiplication](image)

1. **Front end multiplying**  
   \[
   73 \times 8
   \]

2. **Using compatible numbers**  
   \[8 \times 5 \times 10 \times 3\]

3. **Thinking money**  
   \[
   \begin{align*}
   &86 \times 5 \\
   &86 \times 50 \\
   &86 \times 25
   \end{align*}
   \]

*Figure 8. Mental Math Multiplication Problems (Fay’s Chapter 3 Lesson Notes)*

Additionally, Fay often asked her students to relate their answer to the question being asked in the problem and make sure that the answer clearly shows what the question wanted it to show and is reasonable. In other lessons, she stated her goals in terms of mental math and application of math.
concepts as a way to practice sense-making. When I asked Fay about her goals for a lesson on rate, ratio, and percent, she said that one of her goals was to do mental math with percent, adding, “I also want students to understand the real life applications of percent, such as tipping, buying a house, buying something on sale, etc.” (pre-observation interview, April 21, 2015). By having students think about percent in terms of tips and sales, she was able to generate a discussion about the reasonability of the tip or discount, thus developing PSTs’ number sense skills.

In order to decide which topics to emphasize in a 10-week mathematics content for teachers course, with a focus on geometry, Sal drew on his knowledge of foundational mathematics, including the work of Euclid and ancient geometry, and Hilbert and modern treatment of ancient geometry. Drawing on this knowledge his most essential goal consisted of, “Firm up the foundation and revisit some of the more classical things is geometry, such as the axioms, and we proved our first theorem—construction of equilateral triangles” (pre-observation interview, February 3, 2015). In practice, Sal took time to illustrate every proof with models and drawings to accompany the words, in the style of ancient mathematicians, such as Euclid and Archimedes.

Tina drew on her knowledge of the multi-step learning process to structure learning experiences that go from concrete to the abstract in order to build students’ understanding with the help of a physical event. The physical event is followed by a drawing to model the event, then an informal description of the event, followed by a description of the event in a more structured manner, and ends with the use of a symbolic representation. Sometimes, Tina connected the steps explicitly, explaining that the stacking of the cubes was a physical event, in which students learned about leveling off in an effort to understand the concept of average. Other times, Tina created a physical event (such as playing a game to introduce multiplication of decimals) without planning it because the process and ability to structure learning experiences that fit the multi-step learning process are an innate part of her knowledge base. The overall goal of using the multi-step learning process was to provide PSTs with access to mathematics in a way that is suitable to their needs. Some students may need to enter the conversation at the concrete level and work their way up to symbolic, while others may be comfortable with symbolic representations and need
help drawing a picture or a model. The overall experience was meant to address Tina’s overarching goal of positioning mathematics as a sense-making discipline. Most of her instructional moves, including having students model their thinking process, as well as justify it orally and explain it in writing, linked back to the sense-making goal.

In addition to instructional moves, educators’ choices of activity structures are also closely related to their goals and ideologies. Based on the MTEs’ ideological beliefs about who controls knowledge development in the classroom, he/she will make decisions about who presents the information to the class. MTEs who believe students can generate knowledge, rather than merely consume it, will ask PSTs to present their solutions to the class or structure discourse in way that accounts for the knowledge that PSTs bring to the classroom. As can be seen in row one of Table V (MTE’s Activity Structures Linked to Their Goals and Ideologies) on the next page, Alec is a student-centered MTE, who values students’ solution methods and strives to activate students as teachers for each other, instead of claiming to be the only one who can teach the PSTs mathematics. In the quotes from interview transcripts and filed notes presented in row one of the table, I want to show that Alec’s belief—that the knowledge students bring to the class is valuable and must be shared with everyone in the class—is evident in his planning, delivery of instruction, and reflection on the lesson.
<table>
<thead>
<tr>
<th>MTE</th>
<th>Goals</th>
<th>Ideologies</th>
<th>Practice: Activity structures</th>
</tr>
</thead>
</table>
| Alec | Engage PSTs with multiple solution methods; Activate PSTs as resources for each other | Student-centered; Sees students as having valuable knowledge. | • In terms of practice, the class will continue working in groups to solve the problem. And there will be at least 3 volunteers presenting solutions in an effort to help others see other options for solving the problem (pre-observation interview, April 8, 2015).  
• Alec to the class, after a student wrote her solution on the board: “Questions for her? That is one visual take on the problem” (classroom observation, April 8, 2015).  
• What I might do, if I really want to scare them, June has the algebraic solution in all of its messy glory. I might have her present it, and then the class will be puzzled. But it will be important for them to see that solution (post-observation-interview, April 8, 2015). |
| Tina | Develop critical thinking and problem-solving abilities | Mathematical knowledge is a tool for social improvement and everyone should have access to engage in mathematics. | • Notice I never really went to that board and told them how to do anything. I was wise to not do it because to have them investigate on their own they came up with as much information that they needed to know (post-observation interview, June 4, 2015).  
• This is part of Kolb's experiential learning, where you engage them in an experience that they draw on it by writing about it. They think about the abstraction around it (post-observation interview, May 21, 2015). |
| Fay | Have PSTs develop conceptual understanding through teaching each other and small group discussions | Students’ ideas are important and they share the teaching and knowledge developing responsibilities with the instructor. | • When I initially do my course syllabus, I don’t stand up there and read the syllabus or go through it. I give them what is called a scavenger hunt. And I make them talk to each other and figure out the pieces. So I give them a set of questions and they have to search through the syllabus and find the key ideas, and then they talk about it. And I say, “So, as you can see, I am not going to be the one up here who is always telling you things. I am going to have you discussing it, and so your ideas are important” (post-observation interview, February 5, 2015).  
• Obviously, I don't want to take away from the class being content-focused, but I feel if they have to explain something to their peers or create an activity around explaining or introducing a concept, if I think they'll remember it better and understand it better than me explaining it (final interview, May 22, 2015). |
| Sal | Engage PSTs in activities that simulate what a student may feel while learning mathematics; Activate PSTs as resources for each other | Encouraging individual solution paths; Students’ ideas are important and they share the teaching and knowledge developing responsibilities with the instructor. | • We are going to pretend for a moment that we don’t know this thing that we actually do know. Proving new results, there is no ambiguity like that because they don’t know the result so you do something and you get here and you’re like, “Oh ok,” but this was a case where they obviously did know the result. What do you mean they are not allowed to use it yet? Training them out of that in terms of thinking about mathematical proofs is the goal (post-observation interview, February 10, 2015).  
• Oh that was great because there was a rumble in the room. It was not just one person who needed that question answered. So the fact that that slowed it down for everybody's benefit was great, and to have one of their peers talking to them that probably is better because somebody had even vocalized that “Yeah, but you just see the whole thing.” I am like, “Well there may be something to that, so ok let’s slow down, let’s have one of your classmates do it for you” (post observation interview, March 3, 2015). |
MFET 1: Marbles!

<table>
<thead>
<tr>
<th>Number of marbles in bag</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penny had a bag of marbles. She gave one-third of them to Rebecca, and then one-fourth of the remaining marbles to John. Penny then had 24 marbles left in the bag. How many marbles were in the bag to start with?</td>
</tr>
</tbody>
</table>

*Figure 9. Marble Problem (Instructional Handout from Alec’s Lesson)*

On the other hand, MTEs who feel that their job is to transfer knowledge to students in the form of tips and best-practices may choose to present material to the PSTs in a way that does not promote exploration or alternative solution methods. Although none of my participants stated that it was their intention to transfer knowledge, they sometimes worried that they do too much knowledge construction and do not give the PSTs enough opportunities to engage in productive struggle.

One of Tina’s goals for her students was to develop their critical thinking skills so they could rely on themselves more, and on her less, for confirmation of their process and answers. To do this, she had to take their thinking from the concrete to the abstract. She knew that this was not an easy task, but she also knew that mathematics is not just for the elite and that given the proper guidance, her students could all have access to mathematics that they need in order to think critically and problem-solve. As shown, by quotes from her interview transcripts, in row two of Table V above, Tina used David Kolb’s experiential cycle process to scaffold instruction, so that students were able to explore mathematics at a level that they felt comfortable with. The design of these activities took time on her part, but the result was that students felt more independent and in control of their education. They felt like facilitators of their own learning. Tina was comfortable stepping back and letting them explore knowing that she had designed the structure to make the struggle productive.

One of Alec’s major goals for using activity structures, such as students working in groups or presenting solutions to each other, was to make the PSTs see that their ideas are interesting and correct.
This approach was meant to build their confidence as mathematics doers. By asking students to present
and explain their unique solutions to the whole class, he was positioning them as knowledge authorities
not just knowledge consumers. In the last quote in row one of Table V above, he stated that he wanted
one of his students to display her algebraic solution to the whole class. In the interview he states that it is
important for the PSTs to see the algebraic solution, but it was also important to have this particular PST
display and explain her solution path in order to improve her confidence in learning and teaching
mathematics and situate her as a capable doer and teacher of mathematics.

Similarly, Fay and Sal wanted to make sure that PSTs felt like contributors of the knowledge building
process alongside the instructor. As seen in their quotes in rows three and four of Table V above, they
were both willing to share the floor with the PSTs when they saw that their classmates would benefit from
an explanation constructed by their peers rather than an instructor. Fay, even went as far as telling the
PSTs that she will not always be the one talking and that their ideas and contributions are just as
important as hers. Sal did not state his goal of having students understand that they are resources to each
other, but he modeled this goal often and spoke about it with me in the post-observation interviews.
During one of the class sessions, one of Sal’s students came in with a practical problem from her tutoring
field experience. She posed the question of how she should have approached the problem to Sal, but he
opened it up to the class instead. When I asked him about the motivation behind that instructional move,
he stated, “It's an opportunity. Remember that you have resources. Don’t get stuck in the moment. There
are books, there are people, there are colleagues. You can’t figure everything out in the moment” (post-
observation interview, February 3, 2015).

This quote about figuring things out in the moment is a nice lead-in to the last component of the
practice category, decision-making. Westerman (1991) describes decision-making related to one lesson as
consisting of three stages. According to Westerman, decision-making takes place while planning the
lesson (pre-active stage), while delivering the lesson (interactive stage), and while reflecting on the lesson
taught (post-active stage). The quality of decisions made in each stage is essential to the lesson’s success,
because “the quality of people’s decision making—in problem-solving, teaching, and most everything
they do—affects how successfully people attain the goals they set for themselves” (Schoenfeld, 2011, p.36). Therefore, decisions are directly tied to goals at every stage. Here is Alec talking about the in-the-moment decision he made to veer off his plans, “The bigger challenge was definitely the fraction division, but they went with me down that road. But I had to revise because I wasn't planning for it to take so long. I thought that they would get it a little bit quicker. That's why I kept throwing more stuff at them, I tried to push them, and I realized, no, they still, it was so far from their reality, that they need, I felt like they [needed that]” (post-observation interview, April 22, 2015). Alec’s decision to add more examples and practice was directly related to his teaching goal of attending to students’ needs rather than covering the material, and his beliefs that knowledge is constructed not passed down. Consequently, because these goals and instructional moves allow students to engage with mathematics for a longer time and share ideas with others, his learning goal of students attaining deep conceptual knowledge of fraction division was also addressed.

Post-active decision-making occurred after the MTEs had time to reflect on the class session or the semester. While all the participants made changes to their practice as a result of evolving goals and developing knowledge and ideologies, the grain size of these shifts were different across the participants. Alec, because of his tendency to follow students’ responses and needs, as opposed to his agenda, tended to employ decision-making to make adjustments to his goals and practice as the lesson unfolded. As illustrated by the quote above, he made a decision to take more time than he planned with fraction division after observing that students needed more time to grasp the concept. In addition, Alec reflected after each lesson and made changes to future lessons accordingly. His conversations with me during post-observation interviews and his writing of the summaries prompted this reflection. In the summary below, he reflected and explained to the students, that because developing conceptual understanding of rational number addition and subtraction can be challenging, he would extend the warm-up activity of this particular class to give students more practice with justifying the need for the common denominator. His two goals of helping students attain conceptual understanding and following their lead, as opposed to his lesson plan, contributed to his decision to spend more time on a topic his students found challenging.
I handed out a sheet that was meant to be a warm up to adding/subtracting rational numbers (fractions and integers in particular). Each question required you to find the sum/difference using the algorithm (or the method that you like to use) and then to model it visually. My guess was that you'd have little to no trouble with the calculation (the procedure) but perhaps a little bit of trouble modeling (the conceptual understanding). I was happily surprised to see most of you using the fraction models we'd already discussed to model those problems. We'll do a little more practice next time since the details can sometimes be challenging (Class summary, March 4, 2015).

Reflection is a major part of Alec’s practice. I would even go as far as to say that having students reflect on their current and past experiences with mathematics is an unstated goal he has for his students. In describing some of the assignments in his syllabus, Alec asked his students to write a reflection after each assessment, stating “an important part of the learning process has to do with honestly reflecting on and critiquing your own work” and asking them to “be open about what you are learning and to reflect upon your successes and struggles” (Math for Elementary Teachers 1 syllabus, Spring 2015).

Tina tended to reflect on specific goals of the lesson, prompted by my post-observation questions. During one of the lessons, when I asked her if the goals for the lesson were met, she replied with “No, no, no. I think for some reason I just got off track, and so the learning goal—I didn’t even get to what I needed to do today” (post-observation interview, June 2, 2015). When I probed for more details, she explained that she felt the students got frustrated when she asked one of them to draw a figure to go with a mathematical statement. The student thought that her request meant the mathematical statement was not correct. By the time she had convinced the PST that her goal was not to make the student feel that her answer was wrong, but to probe her thinking further by illustrating the math with a figure, Tina had gotten too frustrated to go on. She added, “And so something is wrong because for me to get myself in a situation where I get stressed about it and for them to be stressed about it then that means the scaffolding wasn’t there. It means that a number of things that I had control over didn’t happen” (post-observation interview, June 2, 2015). Generally, Tina’s reflection was almost always directed at something she could do better to meet her goals. In recalling a conversation, she had with a friend about the effect of my research study on her practice, she expressed her reflection in the following way: “I said I’m going to miss her because I really enjoyed talking about what I planned. I said, “I could have been a better planner, but at least I got to discuss what it is that I’m thinking about”” (post-observation interview, May 21,
The reflection that came about from our conversations made her think about ways she could get her students to reflect. She decided to use exit slips to have them reflect on the goals of the lesson. When PSTs asked her if I made her use the exit slips, she protested at first, but later said that my presence did inspire her to access the PSTs’ goals in addition to her own.

Sal and Fay used reflection to evaluate the semester as a whole and drive the changes they wanted to make for the next iteration of the course. As an addendum to the final exam, Sal asked his students to write a reflection on their work during the semester. He asked them to consider their goals, their learning, feelings of being prepared to teach, their grades, and what they did to earn them. During our last interview, he told me that he was impressed with the quality and depth of the PSTs’ reflection and that, in turn, made him reflect more deeply about the semester. He engaged with post-active decision-making by considering a different grading system for the course going forward. He felt that the collaborative work the students did was beneficial to them but was not useful in helping him assess their knowledge. In addition, he told me that he felt that the PSTs’ varied levels of initial knowledge should play a role in how they are assessed:

I think the biggest thing that I would rethink is not the content because I think the content is solid. It’s absolutely the fundamentals of what somebody should know, at least with regard to geometry for elementary teachers. I think I’ve got to come up with a seriously different grading mechanism that somehow, it’s got to be like taking much greater account of where they started and how far they got rather than just how much they know (final interview, March 18, 2015).

By analyzing his practice, in the areas of activity structures and assessment strategies, Sal realized that although working in pairs and small groups created a sense of community among the students, it also may not have given some of them opportunities to develop their learning to the fullest. Nevertheless, students working together and seeing each other as resources was an important part of Sal’s teaching philosophy, so he knew that he did not want to compromise this. He decided to make changes to the frequency and structure of how he evaluated his students. One idea he had was to evaluate PSTs on their growth and mastery of content and not merely on knowledge development. Sal felt that making changes to the evaluation system would benefit his highest and lowest students and prevent what is called in education “teaching to the middle.” Although he has been a mathematics educator, as well as a teacher
educator, for a number of years and felt that his mathematical content knowledge did not need any improvements, he recognized, through reflecting on his practice and his students’ evaluations, that going forward he needed to adjust his goals, knowledge, and practice. Another important shift is that Sal is no longer only thinking about the mathematics his students need to learn, but is also thinking about the mathematics that his students have to learn. Adding the focus of students learning to the focus of content learning created a shift in his ideology from merely content-centered, to content and student-centered.

Fay felt the most confident about the alignment of her goals, knowledge, and practice. During most post-observation interviews, she replied with the cheery “yes” to my question of whether her goals were met. However, upon reflecting on the semester as a whole, she had the most considerable changes to make for the next time she taught the course. Before I explain her new goals, I want to describe Fay’s teaching style. Her classroom reminded me the most of the typical high school classrooms I have experienced and observed. She opened each lesson with answering homework questions and by having PSTs tell her which problems were confusing so she could do them on the board for them. This was followed by a short lecture on a new topic, practice time in pairs or small groups, and assignment of homework. She rarely deviated from this structure and most days accomplished exactly what she said she was planning to accomplish in the pre-observation interview. At our final interview she relayed her new goals to me: to have students talk about mathematics more and to start every class with a problem that students worked to solve, which tied into that day’s lesson’s main topic, and was completely or partially addressed by the end of class. These two changes were significant because they meant Fay had to make major changes to the deeply established organizational structures of her class.

Given that her instructional style was the most unwavering, what brought on these aspirations to completely revamp her goal, practice, and in some cases, knowledge? The answer is reflection. During the semester I observed her teaching, Fay was working on her secondary teacher certifications and taking classes towards this certification. In these classes, she engaged with modern-day research about best practices for learning. She explained her motivation to add microteaching as a component of her class for next semester in the following way:
To me, it's not only that they know the content, obviously, but I wanted them to be able to explain it to someone else, and that's going to reinforce their own content knowledge. But it's also going to tap into, obviously, pedagogical knowledge because they're learning pedagogy at the same time, a little bit of it. I mean, just enough to give them a flavor. Obviously, I don't want to take away from the class being content-focused, but I feel if they have to explain something to their peers or create an activity around explaining or introducing a concept, I think they'll remember it better and understand it better than me explaining it (final interview, May 21, 2015).

The development of new knowledge from her readings prompted Fay to develop new learning goals for her students that would require major changes to her instructional moves and activity structures. In addition to adding a microteaching component to her class, she also decided that instead of her going over the homework for the PSTs, she would have them discuss and make sense of homework problems with each other in small groups. This change to her activity structures was also influenced by her coursework. One of her instructors modeled this strategy by having students discuss homework assignments in groups. Fay recalled that people who did not do the readings were visibly uncomfortable and embarrassed by not being able to participate in the group’s discussion. She was hoping that this technique, of being responsible to your peers, would work in her classes, as well.

Lastly, about a week before the semester ended, the Dean of the College of Education observed her teach a class, and afterward told Fay that her instruction geared towards students with the least mathematical knowledge and understanding, not the range of students with a variety of skills and understanding which sat before her. The discussion with the Dean, following her observation, inspired her to try a problem-based approach to teaching the course. She was however, very hesitant about doing so. Although she has developed the knowledge, goals, and skills to build a problem-based approach course, she lacked the confidence that her students would be able to keep up. She told me:

She really wants me to come out of the gates so to speak, challenging them at the highest level possible first and then saying, “Okay, how can we break this problem down or how can we break this scenario down and bring in the skills we need . . . .” I mean, I cannot . . . I feel like I can't go to that level and be like, “Oh, everybody has to be here” when not even certain people in the class can even . . . they don't even know their multiplication facts (final interview, May 21, 2015).

Knowledge of her students, as a component of her knowledge-in-practice (she has taught the course for fifteen years), gave her a perception of her students’ abilities as lacking, and this did not allow her to fully commit to the goal of taking a problem-based approach to teaching her course. She was worried that
she would scare students away from her course, mathematics, and the teaching profession. She feared that without the foundation she built through direct instruction, students would struggle with the other mathematical content courses in the sequence.

Her fears are driven by her experience, and experience is a major factor that influences decision-making. Mathematics teacher educators can use their experience to anticipate students’ challenges and responses, reflect on lessons taught, and facilitate students’ discourse effectively. On the other hand, experience can make it difficult to abandon certain aspects of knowledge and practice that are no longer useful for meeting new goals. For this reason, effective teacher educators often engage in professional development activities in the form of self-study, learning in communities, going to conferences, and so on. When these experiences are effective, teacher educators are able to strategically revisit their goals in order to consider whether they are relevant and contemporary, decide if new knowledge needs to be developed, and make a plan for how the shift in their goals and knowledge will impact their practice.

Discussion

Teacher educators consider their knowledge and practice, as they plan, deliver instruction, and reflect on their teaching. These considerations are often influenced by current reform topics or trends in education. For example, when the attention to students’ growth and fixed mindsets is a trend in education, teacher educators must familiarize themselves with the relevant literature and decide how they will change or improve their practice accordingly. When the local school district goes on strike, teacher educators must familiarize themselves with each side’s perspectives, and bring the topic into their classroom. But how often do teacher educators think about their ideologies as a tool for reflection? How do their ideologies affect how they address mindsets and teacher strikes in their courses? To their students, teacher educators stress the importance of setting goals, writing clear objectives, and having a strong teaching philosophy, but how often do they themselves examine their teaching philosophies by looking deeply at their goals, ideology, knowledge, and practice?

The relevant literature I read and the data collected for my study inspired me to dig deeper into the question of what teacher educators need to know and be able to do. The result is a framework for
studying, analyzing, and improving practice by focusing on its relationship to one’s goals, knowledge, and ideologies. Next, I will present two case studies and speak to how the components of the framework are used to look deeply at the work of these two MTEs. I have chosen two MTEs who, by their own admission and my observations, made the most of reflecting on their goals, knowledge, and practice, guided by my pre- and post-observation interview questions. Their cases are especially interesting because of the differences in their approaches to being reflective educators. Alec set goals for his students before each class session and revisited these through the course of the lesson, and afterward, by writing class summaries, which he shares with his students. He is reflective by nature and the framework was a natural fit for a number of structures where he already has a place to reflect on his own teaching. Tina’s case is representative of the usefulness of the framework for the study and professional development of MTEs. She explained that due to her extensive experience and expertise, a lot of the decisions she made about her practice were automatic, and being a participant of the study took her reflection to a whole new level. She knew that her practice was goal-oriented and her knowledge was relevant, but she did not consider these in a structured way before. She took advantage of being a participant in my study to review some of the video from the observations independently in order to reflect on her practice. In describing the data collected and analysis of these cases, I want to show how versatile the framework can be as a lens for study of MTEs, but also in helping teacher educators analyze their work.
CHAPTER 6

GOALS-KNOWLEDGE-PRACTICE: THE CASE OF ALEC

Introducing Alec

Alec is a white male. He is an instructor at an urban two-year community college. At the time of my observations he had been teaching the mathematics content course for elementary PSTs for nine years. Before that, he taught mathematics to undergraduate students at various colleges and universities, and before that, he volunteer-taught in a sixth grade classroom for two years as part of an alternative teaching certification program. He holds both undergraduate and graduate degrees in mathematics. Alec is the only faculty member at his institution to teach the two-course sequence of the mathematics content courses for pre-service elementary teachers. As a result, he had a lot of autonomy and decision-making power when it came to the content and curriculum of the course. Out of all of the institutions I used as sites for data collection for this research study, Alec’s was the only institution that, because it was a community college, did not offer a methods-for-teaching course. Alec has a deep knowledge of mathematics, but does not hold a degree that pertains to teacher educating. In addition to his degrees in mathematics, he did graduate level education course work for the alternative teaching degree. He is currently pursuing a doctoral degree in Mathematics Education. Throughout much of his school career Alec tutored his friends and other students in mathematics. He attributed his work as a tutor with being seminal in various aspects of his practice, such as attention to individual learning needs and having a student-centered disposition.

I choose to present Alec’s case because of his natural disposition towards reflection. His reflection process was structured and intentional. This approach to reflection naturally shaped his analysis of goals, knowledge, and practice. His versatile knowledge base stems from his work in both pure mathematics and mathematics teacher education. This versatility was evident in the balance of his goals and practice with respect to teaching mathematics and teaching teachers. As he planned each of his lessons he thought about the mathematical content in relationship to classroom practice. The lessons he delivered contained connections between the mathematical content and pedagogy. Nevertheless, the course remained a content course, and not a methods course. After a lesson on multiplication, where Alec used video of real
classroom instruction to show PSTs methods of multiplication other than the standard algorithm, I asked him how he navigated the balance between content and methods. He replied:

In the content course, students should understand the different methods. They should be able to use them. In the methods course they should understand and be able to use and be able to demonstrate and navigate and troubleshoot. Which, I guess, like we do, troubleshoot. I feel like there's so much overlap in these parts of the course. It almost seems silly to me that these parts of the course are separated (post-observation interview, March 25, 2015).

This view of the content course reflected in course goals, together with his versatile knowledge base and reflective practice, made for an interesting case to study and analyze.

Alec’s path to his current position as an instructor of pre-service elementary teachers at a community college is another factor that made him a unique research participant. In his undergraduate degree, his interests included mathematics, philosophy, and music. After the 9/11 attacks on the World Trade Center in New York City, where Alec lived at the time, he felt a calling to volunteer. He did some research and found a faith-based volunteer-teaching program in Chicago. He felt that his passion for and knowledge of mathematics would translate well into teaching elementary school students. Although he was reading the works of Lisa Delpit and Jonathan Kozol on the plane ride from New York to Chicago, and wanted to make the most out of his time teaching at the elementary school level, he was not planning on making a career out of being a teacher. The shift from this notion, that teaching would not be his life’s work to later realizing that teaching is his life’s calling, is what makes him an interesting case to examine.

Interviews

Data collection for Alec began in March 2015 with an initial interview. The initial interview consisted of two parts—one written and one oral. The purpose of the two interviews was to collect information about Alec’s educational and teaching experiences, personal philosophy, and goals for the course. The observations of his teaching also began in March 2015, with each observation being preceded and followed by interviews. The purpose of each pre-observation interview was to learn about Alec’s teaching and learning goals for each class session, the knowledge and resources he would be drawing on, and to learn about the challenges he anticipated would make the lesson less than successful. In addition, during the pre-observation interview, I inquired about his reasons for choosing particular goals and resources and
the impact of the challenges on his practice. During the post-observation interviews, I asked follow up questions about goals, knowledge, resources, and challenges, and also asked questions that arose during the observations, such as, “Why did you respond to the student’s answer in the way that you did?” or “What made you change the problem you initially wanted to use?” as well as others about his delivery and decision-making processes. My final interview with Alec took place in May 2015. To prepare for this interview, I asked Alec to jot down his reflective thoughts as he graded the final exams. On the day of the final interview, I asked him to share these thoughts with me, as well as reflect on the semester as a whole, using questions such as, “What went particularly well this semester?” and “What was most challenging about this semester?” (A full list of all interview protocols can be found in Appendices A-C).

Observations

While observing Alec’s classes I paid close attention to his instructional moves, activity structures, and decision-making process in order to connect these to the goals, knowledge, and resources that he stated he was drawing on. Some of the connections were easy to make because Alec wrote his goals for each lesson on the board at the beginning of class and referred to them often, but others were more difficult to piece together. Alec had a typed-out plan for each lesson but also believed strongly that this plan should be flexible to accommodate PSTs’ questions and other teachable moments that may arise unexpectedly. Being a responsive teacher meant that improvisation was a large part of his practice and there were many times when the post-observation interview lasted quite a while in order for me to gather all the details about why he made the kind of interactive decisions he did and how his knowledge and goals shaped his decision-making.

Documents

In addition to the observations and interviews, the other data for Alec’s case study came from documents. To prepare for each class session, Alec wrote a simple lesson plan, consisting of the topics he planned to cover alongside possible tools and times. These served as initial guides, but were not set in stone. In his own words, “the plan is in flux” (Initial interview, March 6, 2015), depending upon PSTs engagement and comprehension levels. He took pride in being the type of educator who follows his
students’ questions, understanding, and interests, instead of strictly adhering to the pre-written lesson plans. Because the plan was bound to change and Alec was a very reflective educator, he wrote a class summary at the end of every lesson. This way he could update students who were absent about what took place during class, explain his instructional actions and choices to the PSTs, and reflect on the challenges and successes of the class session. In addition to these documents, I also used his syllabus, classroom handouts, and various assessments to converge several lines of inquiry from multiple data sources.

Below is a table detailing data collection for Alec from March 2015 through May 2015.


<table>
<thead>
<tr>
<th>Source</th>
<th>Description of data collected</th>
<th>Approximate length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial interview</td>
<td>Audio recording, written response documents, my notes</td>
<td>2 hours</td>
</tr>
<tr>
<td>Classroom observations</td>
<td>Video recording from 2 video cameras, audio recording from a clip-on recorder worn by each participant, field notes</td>
<td>19 hours and 15 minutes</td>
</tr>
<tr>
<td>Pre- and post-observation interviews</td>
<td>Audio recording, my notes</td>
<td>5 hours and 30 minutes</td>
</tr>
<tr>
<td>Final interview</td>
<td>Audio recording, my notes</td>
<td>50 minutes</td>
</tr>
<tr>
<td>Documents</td>
<td>Syllabus, lesson plans, class summaries, in-class handouts, written assignments, quizzes, exams</td>
<td>NA</td>
</tr>
</tbody>
</table>

**Goals**

Alec stated the learning goals for each lesson in his lesson plan and wrote them on the board at the beginning of each class. He was also a responsive educator, willing to shift goals or add new goals, in
response to students’ engagement with the lesson. Nevertheless, there were some goals that were so ingrained in his teaching philosophy, that they were a part of every lesson, no matter how structured or improvised it was. These goals included engaging PSTs with multiple representations, exposing them to alternative methods, and challenging their perceptions of mathematics and themselves as doers of mathematics.

Alec admitted that, because this was a mathematics course, his primary goal was for PSTs to be able to do the math, but how he got to that goal was very strategic. He called it his agenda. When it came to operations with rational numbers, for example, his agenda included getting PSTs to understand the interworking of standard algorithms but also making them aware of the existence of methods other than the standard algorithm. Instead of memorizing a trick for where the decimal point goes in a product of two decimals, or the rules for integer multiplication, his goal was for PSTs to understand where and why the decimal point is placed where it is and why a product of two negative numbers is a positive number.

By making the PSTs aware of multiple visual models and representations for exploring mathematical processes, Alec was developing his students’ conceptual understanding, so that they would always remember that there are justifications behind everything they do in mathematics. After a lesson on decimal multiplication, Alec told me:

I don't want students leaving this class thinking that decimal multiplication is some magic trick. It is far from a magic trick why we do what we do with the decimal point, but if they leave with the perception that it is something mysterious, then they're going to carry that perception to their own students (post-observation interview, April 8, 2015).

Alec had high expectations for students knowing how to do the mathematics, but he was even more dedicated to providing his students with tools for conceptual understanding of mathematics.

In his syllabus, Alec explained to his students that, although the course they were taking was not a methods course, they were nevertheless learning mathematics for teaching. He stressed that there were two types of mathematical knowledge—mathematical content knowledge and mathematical knowledge for teaching—and neither was sufficient on its own for effective mathematics teaching. Therefore, in addition to the mathematical content goals described earlier, his other goals for the course included
developing PSTs’ pedagogical content knowledge. He worked toward these additional goals by connecting classwork to future practice, providing them with opportunities to examine how different students learn mathematics, and making them aware of multiple forms of understanding students bring to the classroom. He did this by exposing PSTs to authentic student work as often as possible in order to examine the validity of mathematical methods and solutions and show them the multiple forms of students’ thinking. Alec explained to the PSTs that having mathematical knowledge for teaching meant anticipating students’ responses and being able to understand why students think about mathematics in the ways that they do. As he saw it, these skills can be developed by relating the classwork to classroom practice as often as possible by having PSTs watch and examine videos of actual classrooms, give feedback on authentic student work, and role-play in order to practice communicating mathematics.

Alec felt strongly that teachers needed to validate, respect, and encourage student-invented algorithms. He practiced this belief by asking for multiple ways of solving a problem and validating every solution method, given that it was correct. Likewise, he sought out opportunities to stress this view to his students, prospective teachers. After handing back quizzes in which some PSTs’ feedback to a student’s correct, but non-traditional, solution method did not meet Alec’s expectations, he told the PSTs:

We can think beyond this [standard algorithm] if we want, and that’s ok. And, at least in my opinion, that [alternative algorithms] should be something that should be applauded not punished, so if a teacher says to a student, “Do it the way I am showing you, don’t try anything else,” I think that’s a bad thing. That’s where I am coming from (classroom observation, March 18, 2015).

Alec was aware that many of his students have been educated in a manner that made them believe that teachers have all the answers, and thus all the power, in the mathematics classroom. For this reason, he worked to unburden the PSTs from their mathematical baggage of feeling that they must provide students with all the answers by showing them that, when students see that their methods and explanations are validated and valued, they gain confidence in their mathematical knowledge and abilities. His goal was to get his students out of their traditional mathematics instruction-influenced comfort zone when it came to mathematical teaching and learning. In order to do this, he created a space for them to gain confidence in their mathematical knowledge and ability and challenged their perceptions.
of mathematics and doers of mathematics by exposing them to multiple solution methods and encouraging their original contributions.

**Knowledge**

A large part of Alec’s knowledge and practice was influenced by his *two-world* view of teaching, learning, and mathematics. I took note of this during the first phase of my thematic analysis, familiarizing myself with the data. During the second phase, while coding the data, I again took note that the topic of *two worlds* came up across several different sources. By the end of phase four, reviewing the themes, the divide manifested in the following way: one of the worlds, the rational and factual one, made him hold on to his mathematician identity and made it difficult for him to self-identify as a teacher educator, but in the other world, his heart, he has crossed over to the teacher-educating side of his identity. Given his background of pure mathematics degrees and teaching mathematics to university students on one side, coupled with tutoring friends and classmates from a very young age and volunteer-teaching at an elementary school on the other, it was not surprising that he felt torn between the two worlds.

After finding out about his educational experiences and initial interest in a doctorate in mathematics, I asked Alec if, while taking mathematics courses towards his mathematics degrees, he imagined himself in place of his professors. He replied:

> I think I did at some point, but now I'd be perfectly happy teaching undergrads anything up to a certain level. I've chosen not to teach calculus all this time, and it's not for fear of the content. I think it's because I have the perception that students in calculus don't need the same level of teacher-students in the lower level courses (initial interview, March 6, 2015).

To this point, a lot of his formal knowledge, knowledge-for-practice, came from reading the work of scholars who write about mathematics education. Some of the sources of his knowledge that he cited were Deborah Ball, Mark Thames, Hyman Bass, and the researchers behind Cognitively Guided Instruction.

In addition, his knowledge-in-practice shaped his ability to anticipate students’ responses, choose examples, and scaffold activities in a way that has worked well with past classes and that structure mathematical discourse in a way that will allow for participation from most of the group. When I asked him to describe his knowledge base, he replied:
Content wise. There's nothing that I can be asked that I wouldn't be able to figure out and not only that, and now I have such a variety of ways to explain things. That to me is what's really huge. If you'd asked me five years ago or six years ago, or maybe more than that, "Why don't we need common denominators when we multiple fractions?" I may have been able to piece together a little explanation. Now, I can give you six different explanations like, “They didn't exist before.” I can give you this and I can explain it to other people, too. It's not just me scribbling on paper anymore (initial interview, March 6, 2015).

His diverse knowledge base made him conflicted about certain aspects of teacher education. When I brought up developmental courses for PSTs, he replied that the traditionalist in him thought everyone should be on the same path. When I asked him to elaborate on what he meant by “traditionalist in him,” he replied:

I think what I was trying to say is that there are two views of math living in me. On the one hand, I grew up on traditional math instruction and skill-based mathematics with algebra as god. But in my teaching of developmental education courses and my experiences with students, I've moved away from the traditional view of math-for-math's sake and algebra being all about manipulation, mechanics, and drill/skill. Now I try to contextualize, push the conceptual, and entice them into the subject through inquiry (post observation interview, March 9, 2015).

The two-world view of mathematics teacher educators is also felt strongly in the literature. Although the content courses for pre-service teachers may sound like education courses, they are often taught in the mathematics department by mathematicians rather than in the education department by teacher educators (Goodwin et al., 2014; Masingila et al., 2012; McCrorry & Cannata; Zeichner, 2005).

Alec drew from a plethora of sources to plan and deliver instruction in the content courses he taught to pre-service elementary teachers. They will be discussed in more detail later in this chapter, but I will highlight some of them here briefly. Because he has a growth mindset and considers himself to be a life-long learner, Alec is constantly seeking out professional development opportunities. Because his knowledge of human resources (Chauvot, 2009) is vast, he can call on a number of university and school-level educators in order to seek advice about a best strategy for a topic he is about to teach or obtain a sample of authentic student work. His graduate and undergraduate mathematics degrees and education degrees, coupled with a curious disposition, make him well informed in the knowledge area of professional traditions (Perks and Prestage, 2008). He often draws on something he has read in order to justify his decisions in regards to his instructional moves or curricular choices.
Alec exhibits a life-long-learner disposition and constantly seeks out professional development opportunities to improve his practice. He has been a member of a number of mathematics education study groups, through which he learned about the work of Deborah Ball and Hyman Bass. He completed a series of professional development workshops geared at community college instructors, during which he learned about the work of Ken Bain, Parker Palmer, and Stephen Brookfield, as well as curriculum design and assessment. A fair amount of his knowledge for teacher education comes from his former officemate, a fellow instructor, who taught Alec much about using models and representations in mathematics.

His growth-mindset disposition propels him to continually seek out new ways to teach the same courses he teaches every semester. He considers himself a self-learner and explained:

If I'm just looking for more concrete student examples, I'll have to read to find them, and in reading you kind of go down the rabbit hole and you find other things, and that's where I . . . . It's the midnight readings that really help me to develop. If I'm so excited that I can't stop looking at stuff, then I know that it was probably a good path to go down (final interview, May 14, 2015).

Though many of his sources of knowledge are formal—his education, the textbook, learning from his predecessor and his colleagues at other universities—after teaching the course for close to a decade, he has developed the ability to generate new knowledge through interactions with his students, his experience teaching mathematics courses, and post-active decision making and reflection. Knowledge-in-practice develops as a result of experience teaching, reflecting on the experience teaching, and inquiries into practice in order to improve aspects of it. Alec’s earliest memories of his own experience teaching come from tutoring his fellow classmates well before he became a certified teacher. He recalled that he always tutored—through elementary and high school, college, and beyond. While pursuing his master’s degree in mathematics he taught mathematics courses to college students and prided himself on being given the responsibility to design these courses and having gotten excellent evaluations after teaching them. After he got his master’s degree he went on to teach mathematics in other colleges and eventually became a full time faculty member at the community college where he currently teaches mathematics and mathematics for teachers courses.
Finally, Alec is very reflective and much of his knowledge manifested as a result of reflection. Even though he taught the same topics year-after-year, he was continually reflecting on his practice informally by thinking about new and different ways to teach a topic he has taught before and formally by writing class summaries at the end of every class session taught. In his own words, “I just focus on teaching. That's all I care about, it's all I think about, and I write the class summaries as a means of reflecting upon each day and to kind of make things better” (final interview, May 14, 2015). He posted these summaries for his students to see, thus filling them in on what happened in class and sharing his reflective disposition with them. Often, through reflection, Alec found a new strategy or approach to try, and sometimes he generated an original idea, such as this one for operations in bases other than base 10:

I've always thought of addition in different bases as it's related to mod because it's very similar. One day, I was teaching it, and I thought, "Okay. Algorithm for addition and subtraction, we've got kind of these nice phrases like carry the one." That's catchier than borrow or whatever and I thought, "There may be something for bases." I'm teaching, I'm thinking, "Okay. When I'm in base 8 for instance and I add 7 plus 6, I get 13. I know it's 13. I can't get out of that world. How do I figure out what goes down and what goes up? It's really just 13 divided by 8 is 1 remainder 5. Remainder remains quotient carries." I thought that's the coolest thing. And so like these little stupid things that you come up with as you're teaching (initial interview, March 6, 2015).

By drawing on his interactions with students, colleagues, and self-explorations, Alec produced new knowledge that he then applied to his practice.

Knowledge-of-practice is generated through combining knowledge-in-practice with interpretations of ones’ knowledge-for-practice and connecting it to one’s own classroom practice and larger schooling issues. For Alec, this knowledge came from a study group he participated in with other teacher educators from a variety of colleges and universities. Together they analyzed the work of Deborah Ball, Hyman Bass, and others and discussed its application to the courses they taught. This knowledge not being handed down, but being developed in conjunction with others, is what Alec valued most about these inquiry sessions. He felt empowered that he was already drawing on some of the knowledge constructed collectively and excited when he was a part of constructing completely new knowledge as a result of something he had not considered before.
**Ideologies**

Despite Alec’s strong connection to pure mathematics and proof as the primary source of validation of knowledge, his current view of mathematics and knowledge is fairly relativistic. He is confident in his own knowledge but open to others’ views and interpretations as long as they are well supported and justified. Using Ernest’s (1991) classification, his ideology is part old humanist and part progressive educator, both relativistic perspectives. This dichotomy plays a powerful role in his goals, knowledge, and practice. In terms of mathematical content knowledge, he wanted the PSTs to be able to do the math, but more than that, he wanted them to be able to justify and model the algorithms and rules of mathematics. Furthermore, he explained that just having strong mathematical content knowledge was not enough for effective teaching. Alec made it clear, by constantly connecting classwork to teaching practice, and having PSTs practice communicating mathematics to him and each other, that there was another important type of mathematical knowledge, mathematical knowledge for teaching. The progressive educator in him liked to give PSTs the time and space to explore concepts for themselves and make conclusions based on these explorations, but the old humanist was making him question whether he gave them too much freedom and whether they really understood why mathematical processes work the way they do as a result of these explorations. He aspired to be a public educator (Ernest, 1991) and believed in the power of teaching mathematics for social justice, but said he has not been able to truly incorporate this type of teaching into his courses other than through exposing PSTs to critical analysis of reforms and curricula. He felt that the overwhelming amount of mathematics he needed to reintroduce to PSTs did not leave enough time to “push the envelope” (initial interview, March 6, 2015).

Alec was sometimes too self-critical, feeling that he is not progressive enough in his teaching. Although he felt that he was not incorporating social justice into his practice, I felt that his practice and teaching philosophy attended to the four qualities of democratic education: (a) a problem-solving approach to teaching mathematics, (b) students being taught in a way that that provides numerous opportunities to access and process mathematical ideas, (c) students communicating mathematical ideas with care and respect, and (d) students being encouraged to critically evaluate mathematical data for
social and personal action (Ellis & Malloy, 2007). In fact, his goal of challenging PSTs’ perceptions of mathematics and mathematics doers was in line with the democratic teaching practice of challenging and disrupting “traditional and often non-democratic habits of teaching and learning mathematics” (Ellis, 2008, p. 41).

His ideological commitment to democratic education was the reason he saw his students as individuals who developed complex knowledge deeply rooted in their background and experiences. This stance was something that he practiced and preached by reminding his students to be mindful of non-standard student-invented methods for solving problems and to affirm, not dismiss, these methods as long as they are legitimate. In turn, he praised PSTs, when they came up with original methods that were different from his own. This is what he said about a PST’s explanation of her original method for multiplication:

“It wasn’t what I was looking for because it really was just the standard algorithm in different clothing, but it was creative and different. It had to be. If I hadn’t affirmed it, I would have been just completely going against everything I’ve said the entire semester (post-observation interview, April 6, 2015).

A combination of the influence of a radical ethics course professor and being in New York during the 9/11 attacks drove Alec to seek out a volunteer opportunity. He felt that he could utilize his mathematical abilities to help others understand it better. He entered a faith-based teaching volunteer program and taught at the elementary level for two years. Because he came to teaching through a volunteer program, he regarded teaching as service and critically reflected on his practice as a service or a disservice to his students, and potentially their future students. His ideological commitment to teaching-as-service was the reason he took on the lower level non-credit mathematics courses in addition to the mathematics content for teaching courses. He believed that students who struggle the most with mathematics need the highest level of instruction and was angered that the college usually asks adjuncts, with little experience teaching struggling and underprepared students, to teach the developmental mathematics courses. At one point he was the only full-time faculty teaching these lower level mathematics courses, and he felt that he was truly doing right by the students during this time. The notion of personal responsibility and service
motivated him to set high standards for his teaching and his students’ learning since not doing this would be a disservice to them and the future students that will be educated by the PSTs he instructs.

**Practice**

Alec was often torn between the ideal of how he wanted the class sessions to run and the reality of too much material, too little time, and the great responsibility of teaching all the mathematics PSTs need to know. Ideally, he would have the students construct the knowledge as he facilitated from the sidelines. This type of approach to teaching requires significant planning and a deep knowledge of mathematics, pedagogy, and didactics. It was driven by Alec’s goals of PSTs having opportunities to examine a variety of methods and representations and becoming more independent learners and knowledge constructors. Alec had the knowledge needed to execute this type of learning. Moreover, his knowledge was growing and evolving, due to his commitment to teaching using a variety of approaches that provide for many opportunities for PSTs to access and construct knowledge.

Although Alec had a lesson plan for every class session, he was nevertheless ready to alter or abandon his plans in order to maintain his goals of creating a democratic classroom community where students’ questions are given the time and the attention they deserve and student participation and understanding are more important than getting through the material. When his lesson took a turn off its original path, it may have looked like Alec was improvising—and often he was—but his improvisation was a purposeful function of his ideologies, goals, and knowledge. Expert teachers often improvise, but that does not mean that what they are doing is not planned or focused. When improvising, teachers begin with a plan then fill in the details in response to students’ questions, comments, and needs. Borko and Livingston (1989) explain that, “preparation for such improvisation entails the creation of general guidelines for lessons that are designed to be responsive to the unpredictability of classroom events” (p. 476). Improvisation was a large part of Alec’s practice. Encouraging student questioning and then using their responses to guide his actions was a common instructional move for him. Giving the PSTs the space to ask questions and take part in their own learning was a part of his teaching philosophy. “I want to be
interactive. I want to answer every question. I think that's what makes me who I am as a teacher, like I have to do that” (Initial interview, March 6, 2015).

**Eliciting students’ creative thinking**

On March 25th, I witnessed a lesson that embodied Alec’s teaching philosophy and style. This lesson was a good example of the practices Alec fostered in his teaching of content courses for pre-service elementary education teachers and their relationship to his goals and knowledge.

Below is the lesson outline for this class session:
Wed. (3/25):
- Goals
- Justify why multiplication by powers of 10 is “easy”
- Examine student methods for multi-digit multiplication
- Justify the standard alg. for multiplication
- 11:11:15: Quiz 7
- 11:20-11:50: VR 4, watch video clips and discuss strategies, then use strategies to figure out other problems, connect to visual models and distributive property, in groups consider each clip, attempt to find product of related problem using student method and another strategy, share
- 11:50-12:20: Justification for standard algorithm and alternatives, during this will need to discuss multiplying by powers of 10, virtual manipulatives and activity to help for this
- 12:20-12:45: inconvenient truth again and student multiplication errors, in groups
- HW: HW 12, WA 8 (Lesson plan, March 25, 2015).

At the pre-observation interview, Alec elaborated on his goals by stating that he wanted to expose PSTs to multiple ways of multiplying whole numbers and show them the usefulness of alternative algorithms. He told me that he planned to use a string of videos to show examples of real students’ use of inventive and clever methods to solve multiplication problems. Throughout his description of materials and instructional strategies to be used in the lesson, his goal of a student-centered classroom and his ideological commitment to democratic education were evident without him stating them explicitly. The use of his chosen materials (in the form of the videos) and the instructional strategies (such as encouraging PSTs to use creative thinking in regard to multi-digit multiplication techniques) clearly reflected his goal of exposing PSTs to alternative methods of multiplication and making them see the usefulness and power of student-invented algorithms. When I asked Alec about his success criteria, he replied, “It will be less than successful if the discussions around the video fall flat or if students leave
thinking that the standard algorithm is the most efficient, best strategy for every problem and learner” (Pre-observation interview, March 25, 2015).

The enacted lesson followed the plan fairly closely. After taking the quiz, PSTs were asked to individually find two products (6 x 12 and 12 x 12) using any way other than the standard algorithm. This is a common request from Alec, influenced by his knowledge of Cognitively Guided Instruction (CGI), which is an approach to teaching that starts with the students’ understanding and encourages students to use invented strategies and self-selected tools to find solutions to mathematics problems. Alec often asked his students to solve a problem any way they know how or pretend to not know the algorithms. This is also connected to his goal of connecting class activities to future practice. Depending on what grade the PSTs will teach, their students may not actually know the standard algorithm but will still be able to figure out mathematics problems by using invented methods. Although this is a mathematics content course, it is a course for teachers, and Alec felt strongly about PSTs drawing on their prospective students’ knowledge and strategies to build conceptual understanding of algorithms. Presenting, acknowledging, and encouraging alternative methods is one of the ways that Alec challenged the standard mode of instruction where a teacher presents the standard algorithm as the only method and expects all students to reproduce this method without deviating or questioning the process.

After some time of independent work on the two products, PSTs were given the task of lining up in order of their birthdays, a technique for creating groups in the classroom. Alec had three reasons for choosing this particular way of grouping the students—it provided a good randomization of the class, it allowed the PSTs to find out some new information about each other, and it was a mathematically interesting probability activity. In their new groups, PSTs watched a video clip of a teacher leading a number talk with students. Afterwards they were asked to reflect on the following questions:

After watching part 1, explain the student's proposed method for finding the product of 5 x 14. In addition, answer the following questions.

a) Did it appear that the rest of the class understood the method? Explain.
b) What did the teacher do to attempt to help the class (and the student) understand the method?
c) Mathematically, why did the method work? Use the method to figure out 8 x 16, if you can. If not, explain. (Video response assignment 4, March 25, 2015).
PSTs discussed the answers in groups and Alec walked around and listened to the conversations taking place in the groups. He then asked them to find another product using the method proposed by the student in the video, and explained that,

There are other ways to find this product, but if we are following the student’s method this is how she would have done it. And that’s an important point because, as a teacher, when a student presents a method that is different than what we would do, part of our job is to understand that method, and part of understanding is being able to apply this method to other problems (classroom observation, March 25, 2015).

This statement reflected his knowledge of pedagogy, specifically as it applies to teaching teachers. Not only has he asked them to find the product using the students’ method, but he also explained why he asked them to do that. Smith (2003) refers to this MTE skill as the ability to teach meta-cognitively and “articulate their tacit knowledge of teaching, explaining the whys and the hows of their actions and in-action decision-making” (p. 17). Perhaps because he is a reflective practitioner, Alec was very aware of the necessity to explain the mathematics, but also his use of certain instructional moves, to the PSTs. He often did this in the summaries, which he composed after each class session taught and posted for his students to read. In the summary below, he explained to the PSTs why he interrupted their individual work with some direct instruction, and how the models he showed them are useful to them and their future students.

The main goal for today was for us to become more comfortable modeling add/sub of fractions and integers. We used the visual models to attempt to justify various fraction add/sub problems. After a little time, I noticed that many of you were still somewhat stuck. I took the time to work one of the problems out using the models, which seemed to help. Next we looked at adding and subtracting integers (positive and negative whole numbers). We looked at a variety of models, some visual (number line and chips) and some real world (money, temperature, elevation). We used the models as a means of seeing why the rules make sense. Often students are just taught the rules, but without lots of time spent on why they make sense. As a result, students have blind spots with respect to signs that manifest themselves in their high school and college mathematics (Class summary, March 16, 2015).

Next PSTs watched a second video clip of the same classroom. After watching, they were asked to discuss the following questions in their groups:

1) After watching part 2, again explain the student's proposed method for finding the product of 5 x 14. In addition, answer the following questions.
   a) Pretend you were the teacher. What would you have done differently (if anything)?
   b) Pretend you are one of the peers. What question would you ask the student presenting the method?
2) Reflecting on your past experiences with multi-digit multiplication, compare and contrast what went on in the first 2 videos to your experience? (Video response assignment 4, March 25, 2015).

After giving the PSTs some time to talk in groups, Alec took a seat in the back of the classroom and asked students to share their responses to the questions about the video, as well as their experiences learning multiplication as schoolchildren. This was another student-centered technique Alec used to make sure he was talking less and listening more. Later, Alec told me that he often fights the urge to replace discussion with direct instruction, but this time he had a technique for facilitating the discussion and was also offering PSTs a useful strategy for their future instruction. To explain exactly what he was doing and why he was doing it, he told PSTs “to facilitate an across-the-class conversation I am going to sit in the back so that I won’t be directing the discussion necessarily. We will do this organically, or popcorn, as they say” (observation, March 25, 2015). Research has shown that this type of an activity structure invites and supports active engagement from all of the students in the classroom with the core mathematics being addressed (Schoenfeld, 2014). In addition to the rich mathematics in this lesson, Alec felt the need to attend to equity, as well as rigor. He made his classroom equitable by putting the students’ voices center stage instead of his own.

From the back of the classroom he asked students to share their reactions and prior experiences with mathematics. He waited patiently, too long sometimes, the silence extending into that uncomfortable space where someone felt like they had to speak. It worked—as the silence became uncomfortable more students spoke up. With this class, Alec told me, he generally avoided class discussions because of how quiet this groups of students were—cautious not to be wrong or to be put on the spot. Alec explained that talking about past experiences with multiplication was a good topic for a discussion, since students were offering reactions and experiences and thus the fear of being wrong was reduced. Ellis (2008) explains that having students reflect on and discuss their experiences with learning mathematics is a powerful tool in a democratic classroom, of bringing to light the varied traditional and non-traditional experiences students can have with learning mathematics.
One student shared a memory of doing multiplication fact sheets, and Alec asked how many remember doing these fact sheets. Almost all of the PSTs raised their hands. He took this opportunity to ask how many have seen a technique other than the standard algorithm to find a product of two numbers, and only one student said that she had seen a visual method. This seemingly organic exchange displayed Alec’s ability to make decisions in the moment, using his lesson plan as a guide. Although he did not plan his lesson down to the minute, he knew that asking students to share their experiences with multiplication would cause someone to bring up traditional instruction, which would give him the transition he needed to ask about non-traditional algorithms. This exchange set the class up perfectly for the third clip, which showed a student finding two products (6 x 12 and 12 x 12). If these sound familiar, it is because Alec asked the students to solve these same problems at the beginning of class. Thus, this last activity fulfilled multiple goals: to connect PSTs’ mathematical knowledge to their future practice by having them compare their own methods to those of their potential students, to expose them to non-standard methods and student-invented algorithms, to demonstrate an instructional technique of lessons being started and ended in a strategic way, and to pit the non-standard algorithm against standard methods in order to challenge students’ perceptions of mathematics teaching and learning. This last goal is especially important to Alec, who is committed to helping PSTs see that standard and formal is not always best or appropriate. After the lesson, he told me:

I knew listening to their conversations that there were, as there should be, some disagreement. Because they're going to hold onto what they learned, but then hopefully their souls kind of push against it, and they'll be like, "Wait a minute. Look at all these kids doing this stuff, look at Javier working out 12 x 12 in a wacky—seems like a wacky way—but he's quick. The students in the video they were all with the teacher, watching those methods so kind of having that counterargument or the counter-narrative for this is super important (post-observation interview, March 25, 2015).

After the PSTs watched the third video, Alec asked them to find the product of 23 and 4 using the standard algorithm. He gave them a few minutes to work it out then solicited answers as he found the product on the board. Next, he asked them to find the product of 123 and 92. This led to his explanation of how the standard algorithm works by showing them an activity from the textbook, which explained multiplication by zero. He closed the class by showing the students how to find the product of two multi-
digit numbers by using the partial product method, which he explained also served as a way to justify the steps in the standard algorithm.

If it seems that I spent more time describing and analyzing the part of the class that addressed alternative methods of multiplication, as compared to the standard algorithm, this was done on purpose. The two descriptions are meant to represent the time and rigor of the actual classroom segments related to the alternative and standard methods of multiplication. The presentation and discussion around the alternative methods for multiplication took twice as much time and was structured in a way that gave students time to work individually, in small groups, and as a class to understand the mathematical and pedagogical content. The standard algorithm was presented in the last third of the class with very little participation from the students, outside of occasional solicitation of steps or multiplication facts. This instructional delivery choice on the part of Alec can be explained through an explanation of his goals, knowledge, and ideologies.

Alec’s ideological commitment to democratic practice (Ellis & Malloy, 2007) drove him to expose the PSTs to a variety of perspectives in mathematics. Alec anticipated that many of his students may not have seen any other ways to find a product besides the standard algorithm, and he wanted to make sure that the students they teach get a different education, which includes their teachers’ understanding and appreciation of non-traditional mathematical methods. In the post-observation interview, Alec explained that his goal was to validate the mathematics that students bring to the classroom. He said that the teacher can remain a knowledge authority without dismissing the knowledge his students bring to the classroom. This stance on who has all of the knowledge in the classroom was often met with resistance as PSTs felt that teacher knows best and has all of the answers. Thus, Alec worked hard at positioning the standard, mainstream, status-quo algorithms alongside the other methods, not above it. In the lesson on whole number multiplication, described above, he did that by giving ample time and discussion to alternative methods. He dedicated time to the explanation of the standard algorithm for multiplication but closed the class with another alternative method, partial products. The standard algorithm became just one more way to solve a multiplication problem not the only way, or the main event.
At the post-observation interview, I asked Alec about the goal of the entire activity from the independent work the students did at the beginning of class of finding the product any way they knew how to examining the standard algorithm as a class. He explained that he wanted to “push an agenda that alternatives are called alternatives because they're not the ones that society has deemed to be standard. Students can do multiplication in whatever way works for them, and we saw that happen” (post-observation interview, March 25, 2015). He used the videos to show that students don’t have to be taught the traditional way of doing mathematics all the time, and certainly students don’t have to know the traditional algorithm in order to know how to multiply. He used the videos to show that as teachers we can learn much from students and that sometimes we should do just that—let students teach us. Learning alternative algorithms from students enriches the knowledge base of the other students and the teacher. This is what Alec was saying and modeling by exposing PSTs to authentic student-invented algorithms and asking them to share their own invented algorithms with him and each other.

Alec often put students’ thinking front-and-center. This is consistent with the principles of Cognitively Guided Instruction (CGI), which is a component of Alec’s knowledge-of-practice and an influence in his goal-setting and practice. For example, many of his class sessions did not start in the traditional manner of a demonstration of procedures or methods by the instructor. Instead he often started with him asking PSTs to demonstrate their knowledge by solving a given problem any way they wanted. Other times, Alec asked PSTs to consider different models and representations of the same problem in groups. During these group discussion, if he saw that PSTs were engaged in examining the different methods, he made a decision to cut or shorten or postpone other portions of class activities in order to give the PSTs time to discuss further. These activity structures, instructional moves, and decisions are consistent with Alec’s goals of eliciting students’ creative thinking through exploration and his student-centered approach to teaching. Figure 10, below is a transcript illustrating Alec’s goals of exposing students to multiple representations and eliciting students’ creativity all while checking for understanding. To highlight the planned and emerging goals of the instructor and their connections to his knowledge, ideologies, and practice, I represented this lesson vignette in a table format.
[00:01:03.25] Alec: I am going to give you a problem, and I just want you to solve it. I don’t care how; it doesn’t matter. You’ve already done one like this last time, but it’s worth doing another. Let’s say we had 3 1/4 – 2 5/6. Change that, make it 1 5/6. That would be more interesting. Take a second to solve it any way you know how.

Students solve the problems individually. Alec walks over to the first few rows.

[00:03:23.13] Alec: And I should have said, any way other than the calculator. That doesn’t do us much good. In about a minute I am going to write one method. I am curious to see how many of you did it that way. I sort of already can see what you are doing, but you don’t know what each other’s doing. I am going to write the first step of the method that I suspect some of you did, even if I didn’t see you. (writes 1 3/4 – 1 1/6 in a line). Quick show of hands, how many people did this problem by converting to improper fractions?

A few hands go up.

[00:04:10.16] Alec: Good. I am glad. So what did people do then besides these 2 ways? How did people figure this out? Or did we not?

Goals legend:

| Goal 1 (G1) | Elicit PSTs’ creative thinking with respect to subtracting fractions |
| Goal 2 (G2) | Assess PSTs’ understanding of fraction subtraction (procedural & conceptual) |
| Goal 3 (G3) | Get PSTs to see that there are multiple correct ways to solve the same problem |

Figure 10. Transcript 1 of Alec’s Class Session Segment on March 9, 2015
The transcript above was taken from a segment of a ninth-week class session. This was the second part of a lesson on addition and subtraction of rational numbers, and prior to this, PSTs learned how to model and justify whole number addition and subtraction. Alec started the class by stating his goals and writing them on the board. For this lesson, the goal was to justify and model addition and subtraction of rational numbers. To start, he asked the PSTs to solve a fraction subtraction problem, which he wrote on the board. He told them to solve it any way they knew how ([00:01:03.25] mark). He stressed this point by saying a different version of it a few times: “solve it any way you know how”; “I just want you to solve it, and I don’t care how you do it”; and “it does not matter how you solve it” (classroom observation, March 9, 2015). Drawing on his knowledge of CGI principles, which include asking students to solve a problem in any way that makes sense to him, he set a goal of eliciting students’ creative thinking (shaded in black above) and made a decision to assess PSTs’ understanding of fraction subtraction before he showed them the possible solution methods. As they solved the problem he walked around and looked at what they were doing without commenting. In this way he was further developing his knowledge of possible solution methods and assessing the knowledge of this particular group of PSTs in terms of their procedural and conceptual understanding of fraction subtraction (shaded in light grey above). After some time, he addressed the whole class to let them know that he would be putting several methods on the board, and he wanted to see which methods the PSTs used to solve the fraction subtraction problem ([00:03:23.13] mark).

For this lesson, his goal as stated in the lesson plan was “adding and subtracting rational numbers, justifying, and modeling” (lesson plan, March 9, 2015). This goal addressed his students’ understanding of procedural and conceptual understanding of fraction subtraction. Telling the PSTs to solve the problems any way they know how in order to assess their understanding of fraction subtraction (column G2, above) further reinforces his goal. He wanted to urge the PSTs to have computational fluency, and furthermore, he wanted the PSTs to justify and model. He designed the lesson in a way that showed a progression from the pure mathematics to justification and modeling—revealing the hows and whys of the algorithm. In this segment, his goals and practice were driven by his knowledge and appreciation of
the CGI principles. Some of the principles include encouraging students to use any tools they want in a way that makes sense to them, asking students to solve a problem any way they can, and eliciting students’ creativity (Carpenter et al, 1999)). H asked students to work through the activity in a manner that aligned with these principles. The CGI principles are deeply ingrained in his practice, so that even when he does not identify the goals or aspects of his practice associated with CGI in his lesson plans, they are still evident in his instructional moves, activity structures, and decision making.

Because Alec knew this class to be an unusually quiet one, he chose a problem that they could all solve and gave instructions that made the task seem non-threatening. In this way, he was stressing that the final answer is not the point, the method by which they solve the problem is the point. Using past experience teaching this particular group of students as a guide, he did not solicit solution methods from the PSTs but instead asked them to compare their methods to his, then asked if there are others for which he did not account. He told me in the post-observation interview that this particular group was afraid to speak up and say the wrong thing, afraid to make mistakes. As a result, he tried hard to create a classroom culture that would allow them to speak more freely and be less afraid, while engaging in meaningful mathematical discourse. By structuring the activity in this way, he was able to address his goal of exposing students to multiple solution methods (column G3 above), even if most PSTs used the same method. His knowledge of students, based on his experience teaching the course, as well as this particular group, drove his instructional moves for this lesson segment (row 3 above).

Figure 11, below is a transcript illustrating Alec’s actions pertaining to making students’ original strategies, and the thinking behind the strategy, front-and-center.
<table>
<thead>
<tr>
<th>Transcript</th>
<th>G1</th>
<th>G2</th>
<th>G3</th>
<th>Knowledge</th>
<th>Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>[00:04:10.16] Alec: Good. I am glad. So what did people do then besides these two ways? How did people figure this out? Or did we not? [00:04:22.29] Carly: I gave both numbers a common denominator. [00:04:34.02] Alec: What did you choose for your common denominator? [00:04:36.28] Carly: 12. [00:04:36.28] Alec: (writes while she speaks) Good choice. [00:04:38.15] Carly: And then I changed it kind of into an improper fraction. I can’t do 3/12 – 10/12, so I made a 3 into a 2 and then did 15/12 – 1 10/12. [00:04:53.07] Alec: Interesting. Did you do it vertically or horizontally? [00:05:00.06] Carly: That way.</td>
<td></td>
<td></td>
<td></td>
<td>• CGI principles</td>
<td>• Displayed multiple ways of solving a fraction subtraction problem</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• Multiple methods for fraction subtraction.</td>
<td>• Elicited additional methods from the PSTs.</td>
</tr>
<tr>
<td>[00:05:00.18] Alec: Straight across? That’s interesting. So it’s like that method except that instead of crossing out first, you came up with common denominators first, and then you were able to subtract straight across. Whole number, whole number, fraction, fraction. That works? That’s what you did? (Goes over where he wrote 3 and 1/4 – 1 5/6 in a column) [00:05:25.08] Alec: So over here we could have had 3 3/12, 1 10/12, and Carly said, “I can’t take 10 from 3” so she regrouped, she borrowed. Made the 15 . . . we talked about that last time, how maybe the temptation is to make that a 13 but because its 12 equal parts you want to make that a 15, 15 – 10 is 5/12 and 2 – 1 = 1. 1 5/12. [00:03:38.15] Carly: And then I changed it kind of into an improper fraction. I can’t do 3/12 – 10/12, so I made a 3 into a 2 and then did 15/12 – 1 10/12.</td>
<td></td>
<td></td>
<td></td>
<td>• Multiple methods for fraction subtraction.</td>
<td>• Validated original thinking</td>
</tr>
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<td></td>
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<td></td>
<td></td>
<td>• Connected multiple methods to each other</td>
<td>• Connected multiple methods to each other</td>
</tr>
</tbody>
</table>

Goals legend:

- **Goal 1 (G1):** Develop PSTs’ conceptual understanding of adding and subtracting rational numbers through justifying and modeling
- **Goal 2 (G2):** Connect multiple representations of the fraction subtraction process to each other
- **Goal 3 (G3):** Validate alternative methods and ways of thinking

*Figure 11. Transcript 2 of Alec’s Class Session Segment on March 9, 2015*
By watching Alec teach, and talking to him before and after the class sessions, I learned that one of his most important teaching goals was to challenge students’ perceptions of themselves as individuals who are knowledgeable and able in the discipline of mathematics.

When Carly proposed a new method ([00:04:22.29] mark), Alec knew fairly early on that a form of it was already listed on the board, but it was important to him that she felt that her contribution to the discussion was valued (column G3, above), so he encouraged her to explain her method fully. Even when he did acknowledge that her method was similar to one on the board ([00:05:00.18] mark), he did this by showing a connection between the two methods. Furthermore, he continued to make an effort to understand her thinking and credit her with the solution process. To add to students’ understanding of Carly’s method he touched on a possible misconception ([00:05:25.08] mark), thus sharing the knowledge authority with Carly.

Throughout his explanation of her method, and the other ones on the board, he showed PSTs that there is more than one way to start a problem, more than one way to work it out, and more than one way to express the answer, thus validating the different solution methods and answers he saw when he walked around the room while PSTs worked on the problems individually at the start of class. It is important for Alec to demystify mathematics for the PSTs and challenge their perceptions of themselves as doers of mathematics. Many of his other goals, such as deepening understanding of algorithms, learning to justify and model, engaging with multiple representations and manipulatives, and examining the validity of alternative student methods, are designed to meet the broader overarching goals of demystifying mathematics and challenging students’ perceptions of what it means to know, do, and teach mathematics.

**Connecting classwork to future practice**

Alec made sure to connect the work in the mathematics content courses to the future practice of the PSTs. He first addressed this in his syllabus, by stating, “The Common Core State Standards for Mathematics (CCSSM) are now the status quo. They make certain demands on teachers and students and will require teachers to really step up with respect to mathematics instruction” (Math for Elementary Teachers 1 syllabus, Spring 2015). He further stated,
This is not a methods course. It can sometimes be difficult to separate methods from content, but our main focus is mathematical knowledge. But what is mathematical knowledge? Is there only one type of mathematical knowledge? The answer is no. In this course, we will focus on two types of mathematical knowledge: mathematical content knowledge and mathematical knowledge for teaching (Math for Elementary Teachers I syllabus, Spring 2015).

Alec’s focus on mathematical knowledge for teaching is important, because although the course is a mathematics content course for teachers, the quantity and quality of this focus, is up to the individual instructors of the course. The four participants in my study, attended to the mathematical knowledge for teaching to varying degrees. Alec’s student body was unique because not all of his students are aspiring teachers. Because of the community college context, his students do not have the opportunity to take a methods-for-teaching course. Should they decide to pursue a career in teaching, they can transfer the credits from the mathematics-for-teaching content courses to most four-year institutions that offer teaching degrees. However, some students choose to take the mathematics-content-for-teachers courses simply to fulfill a mathematics requirement. The course may fit into their schedule better than another mathematics course, or they may have heard good things about the course or the instructor from one of their friends. Nevertheless, Alec operated under the assumption that every one of his students will become a mathematics teacher. Therefore, in order to develop effective mathematics teachers, he made multiple connections between understanding mathematical content and teaching others to understand mathematical content.

One way Alec addressed the goal of connecting mathematical content to teaching practice was through sharing his own experiences teaching elementary school students. During a lesson on the topic of subtracting fractions, Alec shared with his students what happens if the teacher shows students a mathematics procedure but does not explain why it works:

So regrouping. I’ll tell you a quick story. When I taught 6th grade I taught this exact topic, and I had a little trick for my students about the regrouping. And I didn’t explain why the trick worked of course, which is why I failed miserably probably (lesson observation, March 9, 2015).

It is important to Alec that the PSTs do not think of mathematics procedures as tricks, and so he told them how he taught his elementary school students a trick, and it failed to make them learn the procedure because he did not provide them with a justification of why the trick worked. To avoid this pitfall with the
PSTs, he explained, with numbers, verbally, and with a picture, how to regroup a larger fraction in order to subtract a smaller fraction from it. At the very end of his explanation, Alec once again connected the classwork to future practice by telling the PSTs,

> It’s a little bit mysterious, but it's nice, and what’s interesting about fraction subtraction is both these methods are taught. This one definitely links up more with what we do in whole number subtraction, but students might be a little bit afraid of this process, and so as a result (walks over to the other method) they might do improper fractions. And I can tell you from my own experience in doing math, if it comes down to time, this method is actually, for me, a little bit faster, using improper fractions. In terms of actually finding these differences, or even if we had a sum, I don’t really care what method you use, but I want you to know how to do both (classroom observation, March 9, 2015).

Thus, in a few sentences, he reminded the PSTs that the classwork in a mathematics content course is directly connected to their future work as teachers. He conveyed that it is essential that they know more mathematics than their future students, that the models and justifications used with whole numbers can be used with rational numbers because mathematics is a logical connected discipline and not a bag of tricks, and that learning is individual and their prospective students will have an individual approach to solving each problem based on their understanding and preferences.

**Positioning mathematics as a sense-making discipline**

Illuminating the individual nature of learning mathematics was a goal related to Alec’s ideological commitment to democratic education. This overarching goal appeared in many other teaching goals Alec set for the course, such as exposing PSTs to alternative methods, providing them with opportunities to examine how different students learn mathematics, making them aware of multiple forms of understanding that students bring to the classroom, and challenging their perceptions of mathematics and doers of mathematics. Motivated by the responses on PSTs’ quizzes from the previous week, he set multiple forms of student understanding as a theme for the class, as he stated here, while challenging the PSTs to consider if changing fractions from proper to improper form will affect their calculations:

> Will it affect the answer, the order in which you find, improper first, then common, or common first then improper? No. Is one of them easier than the other? I guess that’s a matter of opinion, right? That’s an important point for today, the theme of the day based upon the answers to one of the quiz questions: easier is a very subjective thing (Observation, March 9, 2015).
After going over the one visual and three algorithmic methods of solving the fraction subtraction problem, Alec added two versions of a non-traditional method called the “cashier method” to the board. Altogether Alec showed the PSTs six methods to solve a fraction subtraction problem, as shown below in Figure 12.

![Figure 12. Six Different Ways to Solve One Fraction Subtraction Problem (Whiteboard picture from a classroom observation, March 9, 2015)](image)

In order to know about the existence of these methods and the process of their justification Alec was drawing on his formal knowledge-for-practice (i.e., degrees in mathematics and coaching by his officemate), his practical knowledge-in-practice (i.e., experience teaching this particular course and experience teaching elementary schoolchildren), as well as his knowledge of professional traditions (Perks and Prestage, 2008) and his knowledge of human resources (Chauvot, 2009). In this particular case, of wanting PSTs to see the rich multiplistic nature of mathematical processes, Alec’s goals were driving the development of his knowledge and practice. He credited his experiences tutoring with helping him to understand and appreciate the individual nature of mathematical knowledge. Tutoring in a variety
of settings made him want to know more about the diverse methods and instill an appreciation of multiplicity of methods in PSTs.

Throughout his explanations of the algorithms, he solicited some answers from the class but kept the discussion moving by answering most questions he asked. The purpose of this was not to get through the problems faster or ignore students’ contributions—I already mentioned that one of Alec’s ideological commitments is to give students access to the mathematics by putting it in their hands to explore and investigate. Thus, he fulfilled his agenda item of showing the PSTs a few justifications of why the standard algorithm for subtracting fractions works in order to move on to the next part of the class, which was having PSTs model addition and subtraction of fractions individually and in groups. He drew on his knowledge of models, which was learned mostly from his former officemate, who had a deep appreciation for models. He reviewed a range of models and also asked the PSTs to solve some integer addition and subtraction problems without providing them with models for this task. He displayed a variety of manipulatives—chips, cubes, and strips—and told the PSTs that he will not pass them out because he wanted them to decide which ones they want to use. Here, his student-centered view of teaching and learning mathematics and his knowledge of CGI principles, one of which is encouraging students to use any tools they want in a way that makes sense to them, were driving his goals of PSTs using manipulatives and engaging in knowledge construction with minimal guidance from him. His insistence on PSTs exploring topics on their own, offering a choice of representations, and willingness to not follow a set plan to engage PSTs in inquiry-based learning, shows his commitment to equitable education which values students’ contributions and fosters inquiry over memorization of facts and rules.

It’s important to note that Alec asked the PSTs to model the problems not solve them. He knew that they know the rules and procedures, but wanted to challenge the stance that mathematics is a collection of facts and rules, by giving the PSTs the physical and mathematical tools to justify the rules and procedures and show that mathematics is a sense-making discipline. From the examples in his practice described above, to the course outcome in his syllabus, which stated that the course will allow his students to “justify everything in elementary mathematics (the how and the why)” (Math for Elementary Teachers 1
Alec’s goal of demystifying mathematics and positioning it as a sense-making discipline was evident through multiple data sources.

Alec brought closure to the lesson by going over some questions from a recent quiz PSTs took. In the quiz he used an activity structure of asking PSTs to give feedback on an authentic piece of student work to push them to grapple with the mathematics itself, analyze students’ mathematical thinking, and consider ways in which this thinking can be supported and expanded. After looking over the quizzes, he was disappointed that the PSTs’ feedback overwhelmingly showed that they did not appreciate or validate a student’s non-traditional method. Choosing tasks is a job that all educators must do, but for teacher educators choosing a task must include the mathematics and the teaching part of the course description. For this reason, the task of giving feedback to students is a rich and appropriate task, because PSTs have to understand what the student is doing mathematically and give feedback that either corrects or extends the student’s thinking. Alec reminded the PSTs that their preference for a solution method should never be pushed onto students because what seems better to a teacher may not appear that way to a student. He emphasized that an original answer that is different than the one the teacher expected, as long as it is “legitimate and not lucky” (observation, March 9, 2015) should be validated not criticized.

This is an interesting perspective coming from him, since he says that his own education was fairly traditional. Nevertheless, even when teaching elementary school mathematics, with very little teacher training, he knew that he had to give students the tools and space to construct knowledge. He called the school-required curriculum restrictive when he volunteer-taught sixth grade, citing that students were literally restricted to a small box as a space in which they had to show their work. He carried this vision of students displaying what they know, without being restricted by the curriculum or the teacher, to his work as an MTE. The underlying goal behind students constructing knowledge instead of him filling their heads with it is to make sure that students gain confidence in their mathematical ability. According to Alec, students cannot gain the confidence in their abilities until they take ownership of their own knowledge, which he encourages by giving them the space and the tools to investigate and explore mathematics in a way many of them have not experienced before. By learning mathematics in this way,
he hopes the PSTs will gain mathematics skills and confidence to teach mathematics as a sense-making discipline to their students when they become teachers.

Discussion

Alec’s case highlights how an MTE’s knowledge and ideologies development impacts his goals and practices. Most of Alec’s formal educational experiences, school and university, made him view mathematics learning as a teacher directed activity and mathematics itself as a beautiful and mysterious discipline that only a selected few are privy to understand. Everything he has experienced through teaching, however, at the elementary school and college level, has made him see the importance of students’ narratives in teaching and learning of mathematics and the power of student-invented strategies. At first, he was hesitant to cross over to the teacher education world from the world of pure mathematics, but after almost a decade of teaching future teachers, Alec has learned that this two-world perspective is an enhancement to his practice not something to overcome. Thus he draws on his deep mathematical content knowledge to create experiences that allow PSTs to see that mathematics is not a mystery but can be rationally justified.

His goals include helping the PSTs to understand the algorithms, not in terms of the steps of the procedures but in terms of conceptual understanding for why the procedures work. Additionally, his goal is to expose PSTs to alternative strategies and algorithms, knowledge of which he developed through his work as a teacher educator. His practice reflects his goals of appreciating and validating non-traditional methods by asking students to solve problems in a variety of ways and incorporating activities that engage students in multiple representations and justifications for the same process.

Many of Alec’s goals included some form of demystifying mathematics or making it look less like magic. While explaining division to PSTs, he told them that, he believed, teaching tricks and rules to students without explaining why they work was deemed as a disservice to students. Because Alec viewed teaching as service, it was important to him that his students service their future students in a way that is equitable providing them with access to the interworking of mathematics.
Perhaps the most remarkable part of how his knowledge, goals, and ideologies shape his practice was his ability to model what he preached. By constantly pushing the PSTs to explain, justify, and question the mathematical rules and procedures and him, as the one who delivers the procedures, he was developing their dispositions towards mathematics as a sense-making discipline. By allowing the PSTs to work in groups to explore mathematics using any tools they saw as useful, he was urging them to teach in a way that elicits students’ creativity. By being reflective during instruction and in the lesson summaries and providing his students with opportunities to be reflective about their experiences learning mathematics, he was cultivating the qualities of a reflective practitioner in them.
CHAPTER 7
GOALS-KNOWLEDGE-PRACTICE: THE CASE OF TINA

Introducing Tina

Tina was the first one in her family to go to college and the only African American student in the mathematics department, as far as she could tell. She was not sure what she should major in, but she liked mathematics and placed into a calculus course her first quarter in college. She explained to me that her positive disposition towards mathematics was a result of her mother’s excellent arithmetic skills and the desire to develop these skills in her children. Tina remembered, “We would play games when we'd go to the supermarket. She'll say, what’s the total going to be? She would get the total, in addition to taxes” (initial interview, March 23, 2015).

While browsing in the university bookstore, Tina noticed that the calculus book was the thinnest out of all her other textbooks. She recalled that the book’s thinness and her confidence in her mathematical abilities motivated her to declare mathematics as her major. However, Tina soon learned that the mathematics she studied in school was significantly different than the mathematics she encountered in college. The calculus course was a culture shock and she struggled to keep up. Fortunately, her instructor engaged and supported her when she needed guidance. She connected her experiences with learning mathematics in college to the current experiences of her students, PSTs. She recalled that a student once told her that his mother was a mathematics teacher and he felt that he was good at mathematics, but her class made him feel like he did not know mathematics at all. To which Tina replied,

You know what, funny you should say that, because when I became a math major, I thought I knew math, as well, and I learned so much. My calculus teacher, he would tell me things, and he'd have us investigating things, and it's like, “This so cool. I never knew this” (initial interview, March 23, 2015).

At the time that Tina was completing her degree, mathematics majors had the option of taking one education course in order to become certified to teach school mathematics. Tina’s decision to take this course did not stem from her desire to become a teacher. She wanted to get a job upon graduation and was
not sure what else to do with her mathematics degree. Because of this, she took the extra course and became a mathematics teacher.

After completing her mathematics degree, Tina decided to get a graduate degree in Inner City Studies. According to Tina, the specific focus of the Inner City Studies program was on underrepresented students, with the goal to produce teachers who would be prepared to teach those students who fall in the lower quartile, academically. This program provided her with a theoretical framework for critical pedagogy, an important component of her work as a teacher educator. By her own admission, her encounter with the critical pedagogy approach made her open to her later work around access and equity in mathematics, as the director of a national grassroots-based mathematics intervention project. After graduating with a degree in Inner City Studies, she secured a job teaching mathematics at an inner-city high school.

After twenty years of teaching high school mathematics she went back to school and earned a Master’s Degree in Secondary Mathematics Education. She had not planned to go back to school, but the director of the university program asked her to recruit teachers from her school. When there was one spot left, Tina decided to take it. She credited this degree with being the driving force behind her development of conceptual understanding of mathematics. The graduates of the Secondary Mathematics Education program had an opportunity to teach mathematics at the university for a year. One of Tina’s friends had planned to do that, and Tina had planned on going back to teach high school mathematics. Due to tragic events, Tina’s friend was not able to teach and the director asked Tina to request a one-year sabbatical from the board of education to teach mathematics at the university level. As a college mathematics instructor, Tina taught remedial math courses and calculus.

After the one-year teaching assignment ended, Tina applied for a position as a director of a national grassroots-based mathematics intervention project. Initially, she took the job in order to be closer to home and spend more time with her children. She did not know then that her experiences as the director of this project would end up becoming a turning point in how she viewed teaching, learning, and mathematics. After two years, she left the director position with the intention of resuming her career as a high school mathematics teacher. Before she could do that, she got a call from the university, where she had
previously taught mathematics for a year, and was asked if she would come back to teach in the education department. Tina remembered,

> When I met with the dean. . . . She’s this huge figure in the history of education. She said to me, “Here’s the deal. I want you to apply for this professor position.” I gave her every reason why I couldn’t do it. Like, “No, I’m not a mathematician. I haven’t really had a lot of education courses” (post-observation interview, May 2, 2015).

The dean, however, looked at Tina’s experiences of twenty years as a high school mathematics teacher and two years as the director the mathematics intervention project and convinced her that these experiences gave her the beginning knowledge base of a teacher educator, and she could develop the rest on the job. That was over twenty years ago. Since then Tina has taught content and methods courses for teachers at the elementary, secondary, undergraduate, and graduate levels.

Tina’s case is representative of a teacher educator who has come into her role after a lengthy career as a schoolteacher—a high school teacher in Tina’s case. Tina has been an educator for 43 years. She has taught high school mathematics for 17 years and university-level mathematics and education courses for 26 years. For two years she served as a director of a national grassroots-based mathematics intervention project, whose goal was to increase the number of underrepresented students in advanced high school, and college level mathematics courses. It would be easy to assume that someone with such a large number of years of experience and such an extensive knowledge base would be set in her ways and resistant to reflection and change. On the contrary, Tina is continuously improving her knowledge base. “Did I tell you I went to Thailand to present?” she said during one of our interviews. “I’m going to Georgia next week to meet with some mathematicians about the work of the national grassroots-based mathematics intervention project,” she told me during another interview. There was no lack of opportunities for Tina to expand her knowledge base. She did, however, lack a structure to turn gained knowledge into new goals and instructional moves. The pre-and post-interviews of the research study had the unintended consequence of creating this structure for her, and thus she was able to reflect effectively and make changes to her practice based on her evolving realizations and her students’ needs. Tina explained that many of her decisions are automatic and that having the experience of being a participant of the study
took her reflection to a whole new level. Thus, the framework was nuanced by her complex knowledge and ideologies and was validated by her admission of its value as a powerful tool for professional development. She said,

> It helps me to think about how I’m engaging students what I’m using, what my anticipated challenges might be. Because, although it’s in the back of my mind, until you gave me those questions, and I actually sat down, and I had to communicate that to you . . . . Now, on my way to teach, I think about that. I have some kind of rationale for whatever we do, but it’s not as clear as it was before, which means that I need to revisit all that (post-observation interview, May 21, 2015).

I thought if someone with her level of knowledge and experience found such value in the study of goals, knowledge, and practice just think of how powerful the framework could be for a novice teacher educator.

To better understand and examine Tina’s case, I collected data from interviews, classroom observations, and documents. A summary of the data collection is presented below.

**Interviews**

The initial interview with Tina took place in March 2015. During the initial interview I learned about her educational experiences in the areas of mathematics and education and her teaching experiences at the school and university levels. In addition, she shared her goals for the mathematics content course and teacher education in general. The observations of her teaching took place twice a week, beginning in March 2015, shortly after the initial interview. I conducted a pre- and post-interview alongside each observation. During the pre-observation interview, I learned about Tina’s teaching and learning goals for each class session, the knowledge and resources she would be drawing on, and the challenges she anticipated would make the lesson less than successful. Additionally, I inquired about the motivation behind her choices of particular resources and the impact of these resources and her knowledge base on her goals and practice. Tina was eager to reflect after each lesson, and our post-observation interviews often started in the elevator on the way up to her office. During the post-observation interviews she reflected on her meeting of the goals and her decisions to implement certain strategies, invoke specific knowledge, and address unplanned events in the way that she did. In mid-June 2015, I conducted my final
interview with Tina, during which she reflected on the goals of the course, the knowledge gained by her and the students, and the decisions to make changes to certain aspects of the course and her practice.

**Observations**

During the initial interview I learned about the knowledge sources and select theorists who influenced Tina’s teaching. Tina explained that while working as the director of a national grassroots-based mathematics intervention project, she was involved in the development of teaching competencies. She added that she was deeply influenced by this work and has incorporated the development of these competencies in PSTs into her mathematics content and education courses for PSTs. While she taught, I paid close attention to her instructional moves, activity structures, and decision-making processes in order to connect these to the development of the three competencies—accurate listening, cultural understanding, and conceptual adaptability. In addition, she told me that she utilized the multi-step learning process, also a product of her work as the director of the mathematics intervention project, to take PSTs’ experiences from physical events to the symbolic representations of these events. Knowing that this was a central component of her knowledge, which impacted her practice, I paid close attention to identifying the components of the multi-step learning process in her practice. The observations concentrated on connecting the overarching goals of building teacher competencies and positioning mathematics as a sense-making discipline, as well as the lesson goals stated in the pre-observation interview, to Tina’s instructional moves, choice of activity structures, and decision making.

**Documents**

Tina used PowerPoint slides to help her plan the content of each class session. These usually followed closely to the structure of the chapter from the textbook she used, Sybilla Beckmann’s (2013) *Mathematics for Elementary Teachers*. I used these PowerPoint presentations, along with the activities in the textbook as supporting documents to demonstrate commutation between her goals, knowledge, and practice. Other documents used for this purpose included the course syllabus, handouts, and weekly quizzes. In addition, relevant publications, which Tina authored, were used as supporting documents.

Below is a table detailing data collection for Tina from March 2015 through June 2015.
Next, I will describe and discuss Tina’s goals, knowledge, and ideologies. A discussion of her practice, as it relates to her goals, knowledge, and ideologies, is embedded in the description and discussion, which follows.

**Goals**

The relationship between Tina’s knowledge, goals, ideologies, and practice is a representation of the range of experiences from her personal life, schooling, and work as an educator. For example, when I asked Tina about her goals for the course, her first impulse was not to speak about content or pedagogical knowledge, but about comfort level. “I think the first goal is to get people at some kind of comfort level. It's not only comfort level with themselves, but comfort level talking to each other about problems, getting things critiqued, because that's really hard sometimes” (initial interview, March 23, 2015). This goal is indicative of Tina’s experiences as someone who was comfortable with mathematics because of her mother’s guidance but also someone who often felt like she did not belong in the positions she held at the university level. Tina knew how important confidence in mathematical abilities and skills as a teacher
are to being a successful practitioner. She aimed to cultivate a comfort level with the mathematics content knowledge in the PSTs but also confidence communicating that knowledge, since she knew how important this skill was for being an effective teacher and for being a part of a learning community with other teachers. “I think that that level of comfort in their thinking about mathematics, their talking to each other about mathematics, their collective vision, and consensus about what things mean—that's number one” (initial interview, March 23, 2015), Tina concluded.

Next, Tina turned her attention to the goal of deep mathematical content knowledge. This goal includes a number of sub-goals, such as developing mathematical knowledge for teaching, becoming a life-long learner, and understanding the progression and real-world application of mathematical concepts. In addition to being able to understand mathematical processes and procedures and being able to assess the understanding of others, Tina stressed the importance of being able to provide students with accurate explanations and definitions. For Tina, the best way to develop this knowledge was to keep up on professional reading of relevant literature in the field of mathematics, as well as participate in webinars and conferences that strengthen practitioners’ content knowledge by proposing new ideas and looking at old ideas in new ways. She believed these to be necessary for mathematics teachers and MTEs and often participated in a number of professional developments herself. This type of life-long learning resulted in deep conceptual knowledge that included multiple ways to represent and justify mathematical processes, as opposed to memorizing rules and procedures. For example, when learning about integer operations, Tina wanted students to get away from using rules without understanding how to make sense of the rules. She explained,

So you can use a rule that a negative times a negative is a positive, but I want you to be able to set up a structure that would demonstrate that. So here I’m adding the same number on one side, subtracting the same number on the other side, and I get to that. And so as we kind if look at it, it’s not a proof, but it is some kind of evidence or verification that this is what happens (post-observation interview, April 30, 2015).

One of Tina’s overarching goals for the course was to position mathematics as a sense-making discipline. This goal includes having PSTs understand how mathematics topics progress and mathematics understanding builds on understanding of foundational topics. This goal aligned closely with the goals of
the course textbook author, Sybilla Beckman, who stressed the importance of recognizing that operations retain meaning across different types of numbers. Consequently, building a strong foundation of understanding operations with whole numbers will serve as a foundation for understanding operations with other number types, such as decimals and fractions.

Throughout my observations of Tina’s practice and our conversations during the pre- and post-observation interviews, a number of other goals came up frequently enough to constitute them as Tina’s overarching goals for her students in the mathematical content course for pre-service elementary teachers. Tina saw mathematics as a sense-making discipline and not a collection of rules and skills. According to Tina, mathematics processes were meant to be understood, not memorized. To her, this view of mathematics was important to being an effective mathematics teacher and something she felt PSTs needed to develop. Tina saw effective teachers as possessing accurate listening, cultural understanding, and intellectual flexibility—the three competencies she helped conceptualize while working with other mathematicians on a national grassroots-based mathematics intervention project. Tina, along with a team of other researchers, designed these competencies based on their observations of effective teachers. Accurate listening is the ability to actively listen in order to understand the thinking, feelings, and behaviors of students in order to respond in a manner that meets their needs. Cultural understanding speaks to the ability to make an effort to understand and appreciate cultures that are different from your own and to challenge others to be open-minded when looking at the world. Conceptual adaptability is the ability to connect mathematical topics to each other and to other topics and experiences in a way that is novel and creates new and deeper understanding. To cultivate conceptual adaptability, Tina’s learning goals for the PSTs during a lesson on division of fractions were to understand the two interpretations of division, as well as connect these interpretations to the multiplicative structure. In addition to incorporating these competencies into learning goals for her students, one of Tina’s teaching goals was to improve as a teacher educator by continuously integrating the competencies into her own practice.

Tina felt that that although she taught a mathematics content course, she must address more than mathematics content. She must teach PSTs how to understand the content conceptually and contextually
and how to explain it to students using a variety of representations; she must teach PSTs how to look at student work and use it to assess their understanding or lack of understanding; and she must teach the PSTs how to use conceptual adaptability, cultural relevance, and accurate listening. Tina did not teach the PSTs specific methods or techniques for planning, assessment, or instruction, since the course she taught was not a methods course. However, because the course was a content course for teachers, its goals stretched far beyond the understanding of mathematical content. Her experiences as a high school mathematics teacher, a college instructor, and the director of a national grassroots-based mathematics intervention project provided her with the knowledge necessary to help the PSTs meet the goals she set for them.

Knowledge

Ideologies

Tina’s knowledge, ideologies, and practice are heavily influenced by a number of factors—her interactions with her mother, who had exceptional number sense; her education, especially an experience with a significant calculus teacher; her work as the director of a national grassroots-based mathematics intervention project; her work as a high school mathematics teacher in a low-income urban inner-city area; and her work as a university-based teacher educator. Tina was a first-generation college student and the only African American student in the mathematics program. She chose mathematics as a major because it was a subject she grew to appreciate and enjoy as a result of the different ways her mother engaged her with mathematics, such as the “guess how much our total bill will be” game. Her mother always won the game, and through these experiences, Tina developed estimation and manipulation skills that allowed her access to other mathematical concepts.

Although, unlike her classmates, she did not take AP calculus in high school, she placed into calculus in college but struggled. Fortunately, she had a professor that would not let her give up. In our initial interview, Tina explained the impact this professor had on her, as a mathematics student and a human being:
When I walked into the professor, and when I was an undergrad, he had to challenge me. . . . He was from India, Calcutta. He would say, "Miss, you can do this. You can do it." He pushed, and pushed, and pushed until I thought that I didn't have an option. I had to do this. That was the cultural understanding where he knew; he had faith. He had the underlying faith in me that you could do it. I was like, "Okay." You take that ownership on (initial interview, March 23, 2015).

She explained that cultural understanding means understanding other’s experiences and environment in order to challenge them and help them meet high expectations. Not only did she take it upon herself to succeed because this professor had faith that she could, but she also took ownership of becoming this type of a person for others. When I asked her to connect her knowledge to the context in which she taught, Tina replied that she is always pleasantly surprised how many students make it a part of their teaching philosophy to tell her that they are first-generation college students themselves. When she taught mathematics at the high school level, she chose to teach in the inner city. She felt that she had a personal responsibility to help students succeed:

These are the kids that I really want to help. That's the part. It's not like I don't want to help anyone else. It's just, people that you see have that need, and it could have been me as freshman or someone I know. The help just needs to be there (final interview, June 11, 2015).

Critical pedagogy became a part of her theoretical framework, but she did not understand this framework clearly until her coursework for a Master’s degree in Inner City Studies. This degree helped her better understand the plight of underprepared city youth and her role as a teacher to help them reach their potential. Her work as the director of the mathematics intervention project solidified her grasp of critical pedagogy ideologies and led to her work around the development of competencies for mathematics teachers—accurate listening, cultural understanding, and conceptual adaptability.

While teaching in the inner city, she was very aware of her students’ lack of skills and access to important mathematical content. Many of her students struggled with remembering the steps and procedures for mathematics algorithms. In some of the schools she taught, algebra was the highest mathematics course available. This work with teaching underprepared students along with her work as the director of a mathematics intervention project rested on the notion that the change needed to effectively prepare underrepresented students must be led by members of that population. These understandings and experiences had a significant impact on the kind of teacher educator she became, specifically on her
approach to elementary teacher education. She understood how important the development of deep and conceptual mathematical knowledge was to the effectiveness of teachers and how detrimental its lack was to the learning of their students. She understood that knowledge is co-constructed and that her students are knowledgeable and capable.

To describe the different aspects Tina’s knowledge, I am drawing on Cochran-Smith’s and Lytle’s (1999) conceptions of knowledge and practice. Although these were written to describe the knowledge-practice relationships of teachers, the conceptions are useful for describing the essential knowledge for teacher educating.

**Knowledge-for-practice**

The knowledge-for-practice relationship rests on the notion that there is a distinct and recognizable knowledge base for teaching, or in this case, for teacher education. This knowledge base consists of subject matter knowledge, pedagogical knowledge, and knowledge of effective strategies. In this conception of knowledge, teacher educators adapt knowledge produced by others for their needs. This type of knowledge is foundational and serves as the beginning of knowledge constructed and developed by teacher educators throughout their careers. Many aspects of this conception position knowledge as being transmitted from expert to novice, or from teacher to student (Cochran-Smith & Lytle, 1999). In describing this knowledge Cochran-Smith and Lytle (1999) state that initial teacher certification tests often assess a large portion of this knowledge. Because no such tests exist for beginning teacher educators, the knowledge-for-practice in teacher education is not very clearly defined. In their study of 293 teacher educators, Goodwin and her colleagues (2014) concluded that according to their review of the literature and the data from their study, the knowledge base for teacher educators is “not yet codified, coherent, or deliberately integrated into the preparation of doctoral students, especially those intent on professional/professorial life as a teacher educator” (p. 298). This statement applies to the teacher educators in my study. Although several of them completed doctoral degrees with the intent of becoming teacher educators, the degree programs did not include courses specific to learning how to be a teacher educator.
Tina admitted that most of her educational experiences with mathematics were procedural, including her bachelor’s degree in mathematics. Although Tina had above-average abilities when it came to mathematics, it was not until she became a teacher educator that she realized that her understanding of mathematics was mostly procedural. Through her high school and college education she had developed mathematical skills but lacked a deep conceptual understanding. For the most part, she took it upon herself to develop conceptual understanding of mathematics by talking to colleagues, attending workshops and conferences, reading, reflecting, and problem solving. In her master’s and doctoral work, she learned about the theories of Jean Piaget and Jerome Bruner. She engaged in conversations with her colleagues and those at other institutions, including Sybilla Beckmann, Deborah Ball, Johnny Lott, Marilyn Burns, Constance Kamii, David Henderson, and others. She engaged in lesson study and traveled abroad to study from educators across the globe.

Her work as the director of a national grassroots-based mathematics intervention project provided her with the knowledge of the teaching competencies and their designated multi-step learning process. The multi-step learning process peaks students’ curiously with a physical event and slowly builds to symbolic representation of this event, thus adding to a deeper understanding of mathematical concepts. The steps of the multi-step learning process are a physical event, drawing of a picture to model the event, using natural language to describe the event, using academic language to describe the event, and coming up with a symbolic depiction of the event.

To illustrate how Tina’s knowledge of the multi-step learning process and her ideological commitment to mathematics access for all students connect to her instructional moves and goals, I include two transcripts from a lesson in which Tina wanted the students to understand the placement of the decimal point in decimal multiplication problems. Her teaching goals for this lesson, as stated in the pre-observation interview were to “get more into multiplication of decimals kind of back into it by having students think about what they did with fractions, and the learning goals would be to have students actually be able to make the connection between the math drawings and, what Beckmann calls, equations” (pre-observation interview, April 28, 2015).
Her overall goals for PSTs’ understanding of mathematics aligned closely with those of a personal acquaintance and the author of the textbook she used in the course, Sybilla Beckmann, in that she wanted students to understand that operations done to different types of numbers are connected, thus an understanding of how to multiply fractions will become helpful to understanding multiplication of decimals. In preparing to teach this lesson, Tina prepared a PowerPoint presentation, which included a slide with questions for the activities she was going to ask students to answer, listed below:

- 5F What is going on in this exercise?
- 5G Why do we place the decimal point where we do?
- 5H Decimal multiplication and area rectangles (PowerPoint presentation, April 28, 2015).

This PowerPoint presentation did not include a slide about the game Tina had students play. This makes me believe that the decision to play the game was made in the moment, that she decided to implement the game as step one of the multi-step learning process, in order to have students gain a deeper understanding of decimal multiplication.

Tina is of the mindset that all students should have access to learning meaningful mathematics. Her ideological views align closely with others who exhibit the public educator ideology (Ernest, 1991), such as David Kolb, whose experiential learning theories she learned about while working for the national grassroots-based mathematics intervention project. In order to help PSTs gain a deeper understanding of decimal multiplication, Tina asked them to play a game in which they used their fingers to represent numbers and raced each other to find the products of the decimals (step 1: physical event). After a period of time, Tina asked the PSTs to change the location of the decimal point and play again. It is important to note that the game was not part of the PowerPoint presentation, which Tina prepared for the class, nor did the game come up in the pre-observation interview I conducted before observing this class session. Originally, Tina had planned for PSTs to complete activities out of the textbook, which would have allowed them to engage in the rest of the steps of the multi-step learning process. However, at the last minute she decided to add the game so that students could fully engage with all of the steps of the multi-step learning process. The game represented the physical event in regard to decimal multiplication, and
the activities from the textbook, along with the debriefing of the game and activity, took the students
though the rest of the steps in the sense-making process. The vignette below showed the end portion of
the game and Tina’s effort to have students connect the game to decimal multiplication concepts.

I present this vignette in a table format to highlight her goals for the lesson segment. A discussion of
the relationship of these goals to her knowledge and practice follows the transcript below.

Tina’s goal was to connect the activity of playing the game to the procedure of multiplying decimals
(column G1, above). First, she wanted students to have an experience upon which they can draw to make
their conjectures for decimal placement. Second, as communicated in the course syllabus, while this
course was not a methods course, she told students, “Strategies for facilitating student learning will be
modeled” (Elementary Mathematics for Teachers II Syllabus, Spring 2015). Thus she modeled an activity
(a game) for learning decimal multiplication alongside the explanation for the mathematical process of
multiplying decimals, which included identifying the placement of the decimal point in the answer.
Originally, Tina had planned to have students work on the activities from the textbook to understand
decimal multiplication. However, before class started she remembered a game she played with her
students and in-service teachers during professional development. She decided to open with the game to
see if PSTs would be able to make the connection to the physical event of playing the game and the
mathematics involved in playing it.
Figure 13. Transcript of a Segment from Tina’s Lesson on Decimal Multiplication: Part 1, April 28, 2015

<table>
<thead>
<tr>
<th>Transcript</th>
<th>G1</th>
<th>G2</th>
<th>G3</th>
<th>Knowledge</th>
<th>Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>[00:47:43.26] Tina: What is the take away from the activity?</td>
<td></td>
<td></td>
<td></td>
<td>• Multi-step learning process</td>
<td>• Asked PSTs to connect an activity (decimal multiplication game) to the concept of decimal multiplication</td>
</tr>
<tr>
<td>[00:47:48.08] Student 1: Not sure.</td>
<td></td>
<td></td>
<td></td>
<td>• Conceptual understanding of decimal multiplication</td>
<td></td>
</tr>
<tr>
<td>[00:47:50.28] Student 2: Figuring out decimal point places.</td>
<td></td>
<td></td>
<td></td>
<td>• Asked questions to assess PSTs’ understating of the decimal multiplication process</td>
<td></td>
</tr>
<tr>
<td>[00:47:54.13] Tina: Not sure, figuring out the decimal places. What else?</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[00:48:05.26] Student 3: Thinking fast.</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>[00:48:06.19] Tina: Fast. So getting some speed, getting some proficiency,</td>
<td></td>
<td></td>
<td></td>
<td>• Conceptual understanding of decimal multiplication</td>
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</tr>
<tr>
<td>fluency with multiplying numbers. You have to multiply the numbers anyway,</td>
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<tr>
<td>and then you figure out the decimal. Some things that can happen is, if we start</td>
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<tr>
<td>with one decimal point behind and the number six, then they see that and two</td>
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<tr>
<td>decimal points behind then it could be . . . the question would be, “Could it be 0.6</td>
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<tr>
<td>or .06?” (writes the 2 options on the white board)</td>
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<td></td>
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</tr>
<tr>
<td>[00:48:44.05] Student 2: It'd be 0.6.</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>[00:48:59.02] Student 4: 0.06.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[00:49:01.02] Tina: It should be 0.06? If I said two behind, and you got 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>as an answer? How do you know that? So it could be 3 x 2, it could be 6 x 1.</td>
<td></td>
<td></td>
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<tr>
<td>[00:49:33.15] Student 5: It couldn’t have been 6 x 1 because you don’t have</td>
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<tr>
<td>six fingers on one hand.</td>
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<tr>
<td>[00:49:36.03] Tina: Oh, you don’t have six fingers, you are right. I'm</td>
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<tr>
<td>sorry. If it's 3 x 2 you could have this (writes .3 x 2), right? If this is the case, it would be .06</td>
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<tr>
<td>as opposed to .60. Ok, so when would it be 0.60? It could be .60 here (points to 0.6 and adds zeroes) because that’s the same as .60 (writes underneath .6). It is the same as .60000, right? Aren’t these equivalent? Alright, it should be a fun game. I played this game with teachers at a workshop about a month ago, and then also in a 4th grade classroom, and it was a fun game. I didn’t use decimals with the fourth graders. I actually just did multiplication, and they thought it was fun.</td>
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</table>
Tina had expressed to me in multiple interviews that she was looking to make the class more engaging in order to have PSTs pay more attention and take interest in learning mathematics conceptually. She explained that she liked to have PSTs talk about mathematics but felt that often she was doing more talking and thinking than they were. This activity was an attempt to make class more engaging and put learning in the hands of the students. At a minimum, they picked up a strategy they can use with their students; at a maximum they made a connection between the procedure of multiplying decimals and the concept of decimal multiplication. Her goal of encouraging students to think and make sense of their experiences, as opposed to telling them what to think, led her to ask them to describe the mathematics involved in playing the game and the patterns they saw in multiplying decimals. Thus this activity of playing the game was guided by her goal of having PSTs engage in sense-making and her knowledge of the multi-step learning process. As she tried to get students to justify their placement of the decimal point, she was met with silence and low murmurs. Since she saw students were engaged with the playing of the game but not the discussion that followed, she decided to proceed with assigning the planned activities from the textbook in an attempt to make a connection after all the textbook activities were complete.

The second vignette is the debrief of the last activity assigned, 5H (pictured in Figure 14 below), which asked students to find the area of a 2.3 unit x 1.8 unit rectangle without multiplying and discuss how decimal, whole number, and mixed number multiplication are alike. The transcript below illustrates how Tina connected the concrete experience of playing a game to the visual justification of the decimal multiplication algorithm.

Here again, I highlight Tina’s goals for the activity, some stated by her and some deduced by me. In understanding the goals she set for her students, I want to bring attention to the complex relationship of instructional moves, goals, and knowledge in Tina’s case.
<table>
<thead>
<tr>
<th>Transcript</th>
<th>G1</th>
<th>G2</th>
<th>G3</th>
<th>Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>[01:14:01:17] Tina: 5H is another one of my favorites. Why do you think it's a favorite?</td>
<td></td>
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<td></td>
<td>Conceptual understanding of decimal multiplication</td>
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<tr>
<td>[01:14:09:06] Student 1: It's got a picture.</td>
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<td></td>
<td>Asked students to present their solution methods</td>
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<tr>
<td>[01:14:09:22] Tina: It's got a picture. Yeah. And I don’t even need numbers. I can pretty much look at this picture, and pretty much tell the answer and that’s the conceptual part. So, how would you look at this?</td>
<td></td>
<td></td>
<td></td>
<td>Asked clarifying questions and restated responses</td>
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<tr>
<td>[01:14:33:04] Student 2: We counted each of those squares as one, and we know that we have two. So one, two (Tina goes over to the board and starts writing the numeric representation of what the student is saying), and then went over to the right and counted eight bars. So we have two sets of eight. And then we took the bars from the bottom left and started counting those. So we took two of them and put them on the top. So now we have three, and we have one extra bottom bar, and then we move that up so that we have nine. Right now we have 3 9/10. So then we started counting the little boxes on the bottom and there were 24 of those. We took ten from there, and then ended up with four and 14 little boxes.</td>
<td></td>
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<td></td>
<td>Modelled activating students as resources for one another</td>
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<tr>
<td>[01:15:39:15] Tina: (Has written 1 + 1 + 1 and 3 + 9/10 + 24 on the board). Alright, so tell me again.</td>
<td></td>
<td></td>
<td></td>
<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:15:40:13] Student 2: Yes, so we have the 3 and the 9/10, but then we can take one from the small boxes.</td>
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<td></td>
<td>Modelled activating students as resources for one another</td>
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<tr>
<td>[01:15:54:06] Tina: Ok, and then you are going to make this one . . .</td>
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<td></td>
<td></td>
<td>Connected a drawing to an equation</td>
</tr>
<tr>
<td>[01:15:58:29] Student 2: So that makes it 10/10, and then we are left with 14 instead of 24 because we took 10 of them.</td>
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<td></td>
<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:16:10:24] Tina: Ok, this is 14 out of?</td>
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<td></td>
<td>Connected a drawing to an equation</td>
</tr>
<tr>
<td>[01:16:12:14] Student 2: Actually 404, because we counted . . . I am realizing as I am explaining this, it's not clear, sorry, because we have 16 on the bottom, uh, I totally forgot . . . um . . .</td>
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<td></td>
<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:16:34:21] Tina: I followed you here (points to the 3), I followed you here (points to the 9/10), so I don’t know, are you guys? So go ahead tell me</td>
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<td></td>
<td>Connected a drawing to an equation</td>
</tr>
<tr>
<td>[01:16:38:12] Student 2: 14 out of 404, because we counted . . . I am realizing as I am explaining this, it's not clear, sorry, because we have 16 on the bottom, uh, I totally forgot . . . um . . .</td>
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<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:16:44:01] Student 2: We ended up with four and 14/404. I don’t know if it is right.</td>
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<td></td>
<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:17:28:07] Student 3: Oh no, I have a question.</td>
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<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:17:29:04] Tina: A question about this one?</td>
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<td></td>
<td>Connected a drawing to an equation</td>
</tr>
<tr>
<td>[01:17:31:15] Student 3: Where is the 404 from?</td>
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<td></td>
<td>Connected a drawing to an equation</td>
</tr>
<tr>
<td>[01:17:38:00] Student 2: We'd like to take back the 404. That was not right.</td>
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<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:17:45:21] Tina: So here is a question. Are you trying to say that you have, when you take the little boxes would be, if you made them all little boxes, so you are saying that the little boxes would be 404?</td>
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<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:18:13:11] Student 2: Oh wait, we do think that.</td>
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<td>Connected a drawing to an equation</td>
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<td>[01:18:17:17] Tina: Oh, so they think that, because the little boxes are representing hundredths, right? Any other thoughts on the problem?</td>
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<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:18:41:21] Student 4: So the first part I did the same as them. The way I thought about it is the big boxes, so the 1 unit x 1 unit, those represent 1, and then the tall rectangles those represent tenths, and so they knew that they had to count 10 of them in order to move one over so they could come up with a third one. So they have a 10 of the tenths, and then they added that on. So then there is 3, and then they still have 9/10 left over, but the little squares represent hundredths And so if you count ten of those than that’s one tenth so you can add that on.</td>
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<td></td>
<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:19:19:22] Student 4: And then there is 14 hundredths left over, so it's just 4.14.</td>
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<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:19:24:12] Tina: And so it would be out of 100 as opposed to 404, because if you look at that then each part of it, so the little ones are going to be out of 100. What was another way you guys?</td>
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<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:19:45:18] Student 5: Do you want like another equation of how we did it? We just took like the big boxes, we just made that one so we did 1 + 1, since there are two wholes. And then there were two 8/10 ones so we added 1 + 1 + 8/10 + 8/10, and then there was the 3/10 one. And then the little boxes, there were 24 of them, and we figured that the smaller boxes would have to come out to 100 so 24/100. So then we added all of those together with a common denominator of 100, and then we ended up with 414/100, and then we just made it into a whole number so 4 14/100.</td>
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<td>Connected a drawing to an equation</td>
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<tr>
<td>[01:20:42:21] Tina: So here is the other thing, and I think this is a valid question. Because when you look at it you've got the one box, you got the other box, you take that away. The section that you are looking at would be hundredths. So actually weren’t you trying to do it? You were trying to take them all and make them smaller boxes, and I said maybe you might not want to do that, because it would be about 400 boxes when you do it. So each one would be like a hundred, right? Is this is the way I am going to do it—since the 19/10 is the same as 190/100, which is going to be, so we got 24 . . . Ok, alright, but this is a very good problem because what you do is they've got the drawing and you are able to represent the equation from the drawing.</td>
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<td>Connected a drawing to an equation</td>
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Goals legend:
- **Goal 1 (G1):** Develop PSTs' conceptual understanding of the process of multiplying decimals
- **Goal 2 (G2):** Activate students as learning resources for each other
- **Goal 3 (G3):** Show students multiple representations of the same problem

Figure 14. Transcript of Tina’s Lesson on Decimal Multiplication: Part 2, April 28, 2015
Tina believed teaching is most successful when it happens in response to learning. The vignette above is a powerful example of this. When the first PST incorrectly identified the denominator of the fraction that represents the smallest units in the figure ([01:16:12.14] mark), Tina not only did not correct the PST but wrote her representation on the board and left it there while other PSTs proposed their solution ideas. By doing this, she was reinforcing a classroom culture in which students’ solution methods are validated and students learn by listening to each other discuss activities in depth, as opposed to the teacher stating the solution method for the whole class. There is a true teachable moment when she asked for solutions to activity 5H and the first student offered up a partially correct solution. Tina listened carefully, modeling accurate listening, a teacher characteristic she values and wants to develop in PSTs. Accurate listening, Tina explained, is the ability to listen to and appreciate the thinking, feeling, and actions of others and fittingly respond in a beneficial manner.
As she listened to the PST explain her solution, she took notes about it on the board and asked clarifying questions but did not offer judgment or evaluation. Even though the final answer was incorrect, Tina wrote it down and left it on the board for the entirety of the class (Figure 16). Tina knew the answer was not correct but felt it was important that the PST who came up with this solution heard the other responses and used them to correct her thinking. Critical pedagogy and co-construction of knowledge are a central focus in Tina’s teaching. In addition to the PSTs learning from her, Tina knows she will learn from them, and they will learn from each other. In one of our interviews she shared with me that her knowledge base continuously develops through paying attention to students’ thinking. She said, “That’s why I walk around, and I see the different ways of thinking, and I’ll sometimes say, “I didn’t think about it that way. That’s really good. That’s a good way of doing it, and it extends my thinking too” (post-observation interview, May 21, 2012). In another interview she shared that she often asked students to help each other when she felt that the way she explained something did not have the desired effect on a particular student. Relaying that there were several students in her office, and she handed them mini white boards and asked them to explain their thinking to a fellow classmate. Treating students as knowledgeable others builds up their confidence, a component of readiness for teaching, Tina knows is as essential as content and pedagogical knowledge.

Figure 16. Picture of the Whiteboard Where Tina Wrote the Solution a PST Offered, April 28, 2015
As other PSTs shared their solutions Tina listened patiently, modeling such concepts as good listening skills, sharing and interpreting multiple representations, and mathematical discourse elements. She did not praise correct solutions or reprimand students whose answers were wrong. She listened to each solution method and then asked if there were other explanations or ways of thinking about the problem addressing the goal of exposing students to multiple representations of the same problem (column G3, above). Finally, she shared her own method and reiterated what she liked about the problem—the picture allows students to see where the answer is coming from and how it relates to the mathematical equation.

On a macro level Tina’s goal was to help students gain a conceptual understanding of the process of multiplying decimals with the help of the multi-step learning process (column G1, above). Instead of the standard way of teaching students’ decimal multiplication, demonstrating how the algorithm worked and asking students to work on practice problems, she chose to do things differently. First she engaged them with an experience by having them play a game. She tried to get them talking about how playing the game connected to the process of decimal multiplication but soon realized that they were not ready to have this conversation yet. This was in part due to the fact that they had not yet encountered a crucial step in the multi-step learning process—using a picture or a model. She directed students to work on the activities from the textbook, which she knew would include a picture of an area model used to show the multiplication of two decimal numbers. After the students engaged with the problems, she asked them to tell her how they interpreted the picture to gain conceptual understanding of decimal multiplication. By doing this she combined the last three steps—intuitive discussion of decimal multiplication, academic discussion of decimal multiplication, and the numeric sentence that represents the situation. This was not a structured way to have students experience the multi-step learning process. It is possible that the process is so ingrained in Tina’s knowledge base that she invokes it often in some form. In the lesson, I described how the structure of the activities Tina chose worked to engage the PSTs in all the steps of the process and helped them gain a deeper understanding of how decimal multiplication works and how answers to decimal multiplication problems are expressed.
The multi-step learning process was one of the many components of her formal knowledge that Tina picked up from working with other scholars, as part of a national grassroots-based mathematics intervention project and adapted for her work with PSTs. In fact, she confessed that, because of her lack of official education for teacher educating, she spent the first several years as a teacher educator making parallels between the knowledge she developed while teaching high school students to the knowledge needed to teach PSTs. For examples, she explained that when she taught about classroom management she would refer to a technique of standing close to a disengaged student without saying a word. After reading about classroom management, she learned that this technique was called proximity. Thus she built up her knowledge of content and pedagogy related to teacher educating by building on her practice as a high school mathematics teacher. She read to develop the knowledge she did not yet have or find a name for the practices she knew how to execute but not yet name. While implementing her newfound knowledge in the classroom, Tina was developing another type of knowledge by reflecting on her teaching and her students’ learning.

Knowledge-in-practice

Cochran-Smith and Lytle (1999) describe knowledge-in-practice as, “knowledge in action: what very competent teachers know as it is expressed or embedded in the artistry of practice, in teachers’ reflection on practice, in teachers’ practical inquiries, and/or in teachers’ narrative accounts of practice” (p. 262). This conception refers to knowledge as a result of experience rather than reading someone’s research or conducting one’s own research. Knowledge-in-practice of my participants was most evident through their description and discussion of potential challenges related to the teaching and learning goals for each class session.

Tina’s anticipated challenges were often related to her goals. For example, one of her goals was to help students see mathematics as a sense-making discipline. In relation to this goal, Tina spoke of such challenges as having students determine the reasonability of their answers and connect answers to the multiple aspects of the problem in order to assess whether the final answer was correct. Because Tina had taught the course a number of times, she anticipated students would ask her to confirm or reject their
answers, and she strived to have students do this work by themselves in order to develop a deep comprehension related to sense-making. Another goal important for Tina was that PSTs would be able to use a range of models to express their solution paths and processes. Tina wanted them to be able to come up with an equation, a word problem, and a table for most problems. Related to this goal, Tina explained that students often had trouble coming up with multiple representations, stating, “One thing that remains a challenge [for students] is to figure out how you come up with a word problem that’s appropriate for whatever numbers. It could be whole numbers, fractions; now I am talking about decimals” (post-observation interview, April 28, 2015).

To address these challenges, Tina engaged students in learning through inquiry. Whether learning about the concept of average by stacking and leveling off cubes, learning about division by exploring what it means to “fair share,” or learning about geometry by exploring geometric software, Tina gave students numerous opportunities to connect to mathematical ideas in meaningful ways. From reading the relevant literature, and years of experience in the classroom, she understood that the best way to have students understand mathematical concepts is to have them construct knowledge. In our first interview, when I asked her about classroom activity structures, she replied, “What I try to do is get them to do hands-on, as much as possible, discover things that they take for granted, just go back and with a fresh lens, look at these, and draw pictures as much as possible” (initial interview, March 23, 2015).

It is important for Tina to acknowledge the different models and methods PSTs use to solve problems. During the lesson she walked around and listened attentively, then asked students to share their unique methods with the class. Afterwards, she often praised PSTs by pointing out that their method was not one she anticipated or had seen before. In addition, when planning, Tina tried to predict the types of methods and responses PSTs may present. She explained,

When I prepare my lessons I kind of have these anticipated responses, good and bad, correct or incorrect. Then I think about those, and I think about those when I go through because I work through every problem before I come to class. Sometimes I work through them, and I don’t really have a check point other than, “makes sense to me” or “this does not make sense” (post-observation interview, June 20, 2015).
Other times, even though she did not think about a certain checkpoint in advance, she decided to check for comprehension during delivery of the lesson. It is important to Tina that students understand that mathematics topics build on and connect to each other.

In the previous examples, influenced by her knowledge of the multi-step learning process, Tina chose to add an activity that was not part of her original plan. This decision related directly to her ideological stance of providing all students access to mathematics through a set of experiences that build up their understanding from concrete to abstract. In the next example, Tina once again altered her plan, but this time she was driven by knowledge-in-practice to address a goal that was not originally in her plan. The challenge PSTs faced, in relation to the learning goal of understanding scientific notation, was not one Tina had anticipated for this lesson. Nevertheless, because of her extensive experience teaching mathematics to pre-service elementary teachers, she made an in-the-moment decision to check for their understanding of decimal numbers during a warm-up problem on the topic of scientific notation.

Just as she did in every class session, Tina started out the lesson by asking PSTs to work on a warm-up problem. This was the sixth week of the course and the main topic for the class was division, but before getting started on division, Tina asked the PSTs a warm up question about scientific notation. She did this because when the class revisited scientific notation during the previous lesson, Tina noticed that changing a number to and from scientific notation did not feel intuitive to the PSTs. Although they were going through the motions by following the rules, they were not making conceptual connections. She decided to start the next lesson by working on selected scientific notation problems with several goals in mind—helping the PSTs gain a deeper understanding of the purpose of scientific notation and helping PSTs develop fluency with expressing numbers in scientific notation. Below is the warm-up problem, taken from that day’s PowerPoint presentation (Figure 17). What follows is the debriefing of the warm-up problem. This transcript illustrates how in-the-moment reflection and decision making is influenced by Tina’s knowledge and impacts her goals and instructional moves.

Warm-up problem:
Place the following ordinary decimal notation in scientific notation –

a) 123.45
After PSTs worked on the problems for about eight minutes, Tina initiated a debriefing of the activity by reading the first decimal from the warm up and asking for a volunteer to read the second decimal. Although the local goal for this lesson segment was to express a number in scientific notation, after reading the first decimal number Tina made a decision to activate an overarching goal for the course—connecting mathematical concepts to each other. She was also activating an overarching practice of modeling instructional strategies by showing PSTs how to connect previous knowledge to the current topic. Since this course was the second in the sequence of mathematical content courses for elementary PSTs, learning how to read decimal numbers was a topic the class engaged with during the previous quarter. By asking PSTs to read the number, Tina was highlighting the connectedness of mathematics—how topics and procedures in mathematics are not a collection of facts, but a unified body of knowledge (Ma, 1999). She was also using her knowledge-in-practice to check for understanding of PSTs’ abilities to read a decimal number. In the post-observation interview, Tina relayed that she was surprised that her request for a volunteer to read the decimal number was met, first by silence and blank stares, and then by incorrect identification of place value. To mitigate their confusion, she modeled another effective instructional technique by asking students to read only part of the number, and then building on this understanding to read the entire decimal number. Although this was not a planned part of the lesson, Tina’s confidence in her content knowledge allowed her to proceed with the unplanned segment of the lesson in order to have students meet the emergent goal of using proper academic language to read numbers.

Throughout the lesson segment, Tina kept the PSTs engaged by asking questions and giving ample waiting time between questions and answers. She could have easily given a mini-lecture on place value and reading rational numbers, but her commitment to critical pedagogy, which embraces students engaging in knowledge co-construction as opposed to knowledge reception, kept her from lecturing. Tina’s extensive content knowledge, knowledge of effective pedagogical practices, and her understanding of mathematical knowledge drove her to involve PSTs in grappling with the challenge of reading a
decimal number. Her goal of activating students as resources for one another prevented her from just stating the facts, and instead, lead her to have PSTs intervene when their classmate needed help. In the process, she reminded them that the knowledge of reading numbers is part of their knowledge base for teaching and should be well developed. Once again, as stated in her syllabus, although the course is not a methods course, one of Tina’s overarching goals is to develop PSTs’ mathematics knowledge for teaching and expose them to instructional strategies that can be used in the classroom.

After the PSTs connected place value to reading the decimal correctly, Tina continued modeling effective teaching strategies by extending the teachable moment to include writing the decimal number as a fraction. By doing this, she was demonstrating the connectedness of mathematical ideas and the importance of multiple representations as a component of mathematical knowledge for teaching. Her knowledge of students and their potential misconceptions and knowledge gaps contributed to this decision. She felt that if the PSTs needed a reminder of how to read a decimal number, then they most likely would struggle with expressing the decimal as a fraction. Her intuition was confirmed when a number of students proposed the wrong denominator. Drawing on her deep mathematical content knowledge, Tina went back to the concept of place value to help students cope with this challenge. She ended this teachable moment by reminding students that reading and expressing a range of numbers is part of the knowledge base for teaching and something they must be able to understand as teacher candidates. By connecting the challenges of reading and expressing numbers to place value, Tina also addressed an overarching goal of moving PSTs from procedural to conceptual understanding—the denominator of the fraction is one-millionths not simply because there are six numbers after the decimal point, but because each of those numbers represents a place value with the seven being in the millionths place.

One of Tina’s teaching goals was to lecture less. Even when the students did the activities, the debriefing was usually led by her. This particular debriefing described in the vignette was led by the PSTs. By being a responsive educator, Tina was able to have students engage in a productive struggle to understand their own misconceptions, see mathematics as a connected discipline, and use multiple
representations. This incident was a learning experience for the PSTs and for Tina. By reflecting on what happened during class, Tina was able to identify several goals for future class sessions, such as spot-checking more often for student understanding of how topics connect to each other and making more explicit the essential knowledge needed for teaching. Setting these goals for herself, instead of blaming the students for their lack of understanding, is characteristic of Tina’s identity as a life-long learner. She saw all experiences, no matter how structured or unexpected, as opportunities for her to grow her knowledge base and improve her practice.

**Knowledge-of-practice**

The knowledge-to-practice relationship embodied by knowledge-of-practice stems from the idea that teachers develop their knowledge base, across their professional life spans, by understanding and analyzing their knowledge and practice and connecting them to the broader constructs of teaching and learning. These connections may encompass school improvement, policy change, or education for the purpose of emancipation. Some of the underlying principles of this knowledge-to-practice relationship are: learning begins with identification and critique of one’s own experiences, beliefs, and stances; knowledge is socially constructed by educators working together in inquiry communities; inquiry is looked at as a larger effort to transform teaching and learning (Cochran-Smith & Lytle, 1999). For teacher educators, knowledge-of-practice entails merging their teaching and learning so that “as they learn in practice, they are also learning of practice” (Goodwin et al., 2015).

For Tina, drawing on knowledge-of-practice (Cochran-Smith & Lytle, 1999) means putting into practice what she learned through reading, experience, and earlier research and reflecting, analyzing, and critically looking at this implementation of what was learned. The national grassroots-based mathematics intervention project, which Tina directed for several years, developed a competency-based program for teachers to learn to lead the professional development of other teachers, groups in schools, communities, or national-level events. Tina was involved in this work and it deeply resonated with her. She used the competency model to reach her goal of developing the PSTs into effective teachers. In telling me about her tenure promotion work, she said, “I went back to the competencies, and I studied about how, when I
do things, I think about being conceptually adaptable, have culturally understanding, and actually being empathetic for people, being an active listener. That’s the way I started, and I framed everything around those three competencies” (post-observation interview, May 21, 2015).

In a chapter she wrote, titled, Making Sense of Students’ Fractional Representations Using Critical Incidents (2011), Tina examined how her understanding of fractions impacted the understanding of fraction concepts of the PSTs in her courses. She described how accurate listening and intellectual flexibility (two of the competencies) added to her delivery and PSTs’ understanding of operations with fractions. Tina marveled that after many years teaching she was still struggling to realize what effective elementary and middle mathematics education courses should be like. She thought it was natural for her to examine her own knowledge and continue to learn, believing that teaching mathematics teachers, like teaching in general, entails continually developing new knowledge and resources for teaching. She felt that her collaborations with faculty in the education and mathematics departments, as well as mathematics professionals who were part of the grassroots-based mathematics intervention project she directed, helped her better address the lessons she learned from the critical incidents she described in the chapter. One of these lessons was the identification of two competencies, accurate listening and intellectual flexibility, as areas that could significantly reinforce her delivery of instruction. This type of research is consistent with the image of knowledge and practice transformed and expanded. As a result of the research that Tina conducted, “what goes on inside the classroom is profoundly altered and ultimately transformed” (Cochran-Smith & Lytle, 1999, p. 276).

At the end of the chapter Tina identified a goal to work on - to generate specific concrete examples to illustrate difficult abstract concepts, use detailed participant data to support her claims, and use analogies or metaphors to illuminate concepts and principals. This is an example of her knowledge and ideologies impacting her goals, which in turn, as I observed, impacted her practice. Although not explicitly stated in the quote above, the idea of generating concrete examples in order to illustrate abstract concepts comes from her knowledge of the work of Jerome Bruner. Bruner (1966) described three stages of cognitive representation—enactive, iconic, and symbolic. The enactive stage refers to the concrete examples Tina
speaks to in her chapter. As this is the first stage it requires tangible examples that are easy to grasp. The iconic stage involves images of the concrete representations. Tina often told her students that pictorial representations were her favorite since they serve as a bridge between the concrete and the symbolic and thus can be elaborated in terms of either or both concrete and symbolic representations. Finally, the symbolic stage is what Tina described as abstract in the chapter. Understanding at this stage allows students to see patterns and make generalizations, which is essential in mathematics.

To explain new concepts or address challenges the PSTs had with mathematics content, Tina often invoked Bruner’s three stages. In reviewing student work samples with the PSTs, Tina explained why one student’s explanation was more effective that another’s by pointing out that the picture in the student’s response was drawn to clearly illustrate the components of the mathematical equation:

Not that A is completely wrong. It's just that when we look at our presentations and how we represent our work, it should actually align with the numbers. So I should be able to look at the picture and tell you what the equation is, look at the equation and tell you what the picture is (lesson observation, April 28, 2015).

The idea of going from the concrete to the abstract in also embedded in the multi-step learning process she uses with the PSTs, which begins building students’ comprehension with the help of a physical event and ends with the use of a symbolic representation.

The overall goal of using the work of Jerome Bruner, David Kolb, and the national grassroots-based mathematics intervention project is to provide students with access to mathematics in a way that is suitable to their needs. Some students may need to enter the conversation at the concrete level and work their way up to symbolic, while others may be comfortable with symbolic and use the enactive and iconic representations to illustrate their understanding. Tina made sure to expose PSTs to a number of representations for them to choose from, so that they can gauge where they felt most comfortable. During the initial interview, Tina stated that, in addition to theories of teaching and learning, she gave consideration to “how teacher candidates can impact the community and increase the number of underrepresented students in math and science areas” (initial interview, March 23, 2015). These
considerations are supported by her work to position mathematics as a sense-making discipline that can be accessed from several different entry points, thus making it within reach for all types of learners.

When I asked Tina about her current research she explained that, in addition to being the edTPA (teacher certification exam) coordinator, she was involved with a research project on the early implementation of the edTPA. She explained,

We're doing research on edTPA in terms of what happened this year, how we prepared people, how we got students to do the pilot, how we're preparing our mandated students for fall, then looking at the scores for this year and kind of comparing it to next year. We're going to do an AERA proposal.

When I asked how her research impacted her practice, she explained that her involvement with the edTPA has necessitated development of knowledge around the academic language of mathematics. She explained,

With the push for edTPA and the academic language, this is something I am becoming more and more involved with. I'm trying to understand the language of mathematics, like the vocab. Not only vocab but the syntax, the function, those kinds of things, and getting our students to understand the differences, and when they can use them.

Furthermore, she encouraged her colleagues to pay more attention to the language of mathematics, thus asking them to develop their knowledge base to accommodate for new developments in education. In addition to learning more about academic language, Tina introduced a new activity structure, examining student products, into her practice in order to give students experience with analyzing student work—an important component of the edTPA. This was a carefully crafted activity where she selected two quizzes from the class and asked students to examine them by posting the following prompt: “Please compare Student A and Student B papers. What do you think about each one? How are they alike? How are they different? Does one stand out more than the other? Why or why not?” (lesson slide, April 28, 2015). PSTs analyzed the two papers and engaged in a discussion afterwards. Although at the end, Tina asked the PSTs about the kind of feedback they would give to the two students, the discussion concentrated on the mathematical content of multiplication, pictorial representations, and area models as a way of connecting the picture to the mathematical equation.
To develop new ways to analyze her own teaching, Tina, with a team of other educators, worked on helping in-service teachers implement Lesson Study in their schools. Lesson Study is form of professional development that teachers in Japan use in order to learn from planning, teaching, and reflecting. The most common form begins with teachers researching the lesson topic by looking at curricular materials and reading relevant literature. The teachers then design and teach what is called a “research lesson.” Other team members observe the lesson, which is followed by a post-lesson discussion.

Together with a colleague, Tina traveled to Japan to document how Lesson Study is done there. She was most influenced by the post-lesson discussion and the contribution by a knowledgeable other, someone who comes in to provide comments at the end of the post-lesson discussion. She learned that “the knowledgeable other is responsible for: (1) bringing new knowledge from research and the standards; (2) showing the connection between the theory and the practice; and (3) helping others learn how to reflect on teaching and learning” (Takahashi, 2014, p. 10). After discussing the Lesson Study benefits with me during a post-observation interview, Tina concluded:

> When you sit there and you actually debrief the lesson, then you’re saying what you were thinking, what your students were doing, how they reacted, how you would do it differently. And then the expert will come in . . . . That could be a key for math educators as well. But what the expert contributes to the discussion, it’s always about listening to what’s going on, but then taking it to a higher level—more for content than pedagogy. It’s that you're moving them to think about math at a higher level and summarize (post-observation interview, June 2, 2015).

Following the trip to Japan, Tina got involved with a local group that worked together to promote and support Lesson Study. Now, when she works with teachers through the national grassroots-based mathematics intervention project, she encourages them to join the group in order to learn about Lesson Study and improve their knowledge and practice.

When I asked her how her trip to Japan influenced her teaching, she replied that she was surprised to see the challenges that the students in Japan had, related to conceptual understanding and sense-making, were similar to those of American students. She said it was helpful to discuss similar concerns with the teachers and brainstorm solutions together. These discussions helped her develop knowledge-of-practice.
Discussion

Tina’s case portrays an MTE whose instructional moves and goals have a strong alignment to a few pieces of key knowledge. Although she is continuously developing her knowledge base by attending workshops, classes, and conferences, she is very selective about the new strategies she adopts, based upon their alignment with her current beliefs and well developed goals for the PSTs. Developing the competencies of accurate listening, cultural understanding, and conceptual adaptability are at the core of her goals and practice. Through activities that allow students to engage with multiple representations, she was cultivating their skills of accurate listening and conceptual adaptability.

The multi-step learning process was the other key knowledge component, which Tina draws on to inform her goals and practice. With the goal of developing students’ conceptual understanding, Tina created experiences that fostered PST’s understanding of concrete and abstract mathematical concepts.

Whether through multiple representations in her instruction, allowing students to revise and resubmit assignments, or making herself available to students outside of class, Tina was continuously working on ensuring access to mathematics for all of her students. Because of the opportunities that were afforded her and the roles she has held and continues to hold in education, she never takes for granted that not everyone has the opportunity to engage with mathematics equally. Providing opportunities for students to engage with mathematics and see themselves as its capable learners was perhaps the most important teaching goal for Tina.

One of the student outcomes of the conceptual framework for the teacher education program at the university where Tina teaches pertains to supporting diversity. It states:

Prospective teachers should:
1. understand that it is essential that schools and communities accept the goal of mathematical education for every child;
2. examine their expectations about what children can learn and do;
3. strive to create learning environments in which high expectations for children are met. (Elementary Mathematics for Teachers II Syllabus, Spring 2015, p. 1)
Tina not only included these outcomes on her syllabus but also incorporated them into her daily practice by designing tasks that were challenging yet accessible, reaching out to students who needed extra help, and modeling cultural understanding in her instruction.

One of the ways Tina met the needs of her students was by constantly adding to her knowledge base. This made her an immeasurable resource to her students. She had a deep knowledge of the Common Core State Standards (CCSS) across disciplines because she was working on a project that analyzes the standards. When I asked her about the value she sees in the CCSS for Mathematics her reply encompassed two sections of her knowledge base. First, she said it was the reality that PSTs would be facing once they became teachers, which highlights her knowledge of current educational situations. In addition, she believed that the standards bring to the forefront what it means to really know mathematics. She explained that a few of her students thought they knew mathematics before they started the class, but they knew it only procedurally, not conceptually.

For most people to get a sense of, yes, we cross multiply, yes, we divide, but then my question was, “Why? What does that give you?” That gets back to the sense-making, and if nothing else and if they can't ask the question, I want them to at least ask the question of themselves. Like, “What does this mean? What does it mean to do this? What does it mean to do that?” That’s something that is lost—or was lost—and I think common core brings it to the forefront and the surface and to say, “You can persevere, but then you also have to do some sense-making, and what does that mean to do the sense-making?” In that respect then people were at least asking themselves those questions. It's always when you ask those questions, they always go back to procedural, and tell you the steps. It's like always the steps you can use. You don’t negate that, you just say, “Yeah, we cross multiply and we divide.” Then just, “Let's go back. What does that really mean?” (post-observation interview, June 4, 2015).

By incorporating the CCSS and Standards for Mathematical Practice (SMP) into her teaching, she was continuously reminding students that mathematics is about sense-making, modeling, justifying, and finding patterns. This knowledge influenced her choice of instructional materials, since Sybilla Beckmann, the author of the course’s textbook, shared Tina’s views about the CCSS and SMP and mathematics as a sense-making discipline. This knowledge also deeply influenced her practice and her goals as evident by the focus of most lessons on the importance of connecting to the standards, the importance of connecting mathematics concepts to each other (for example, distributive property to multiplication, multiplication to division), and the importance of having justifications for rules, thus
bringing understanding to the theory behind the rules. At the core, this knowledge was connected to her ideological beliefs about mathematics teaching and learning. Tina sees mathematics as a civil right for all students regardless of their backgrounds. As such, she believes that all students have a right to a high-quality conceptual understanding of mathematics, not a watered-down, shallow, skill-based knowing of facts and algorithms. To ensure that her students got the mathematical education they deserve, she was continuously working to develop more knowledge that could be beneficial in her teaching. Although she confessed that she did not consciously think about her goals, until I started asking about them, when one’s knowledge base aligns with one’s ideology, the goals and practice line up alongside them.
CHAPTER 8

DISCUSSION AND IMPLICATIONS

I designed this study to examine and learn about the knowledge and practice of mathematics teacher educators. My motivation to learn about the knowledge and practice of teacher educators came from my work as a school-based teacher educator, my personal goal of becoming a university-based teacher educator, and my identification of a limited amount of empirical case studies in the emerging area of research on the knowledge base of mathematics teacher educators. As the study evolved, it came to include the examination of MTEs’ goals, knowledge, ideologies, and practice.

In this chapter, I will begin with my own reflection on this research as it relates to becoming a mathematics teacher educator. Next, drawing on the cross-case analysis and the individual case studies, I will identify key findings pertaining to the goals, knowledge, ideologies, and practice, and the relationships between them, as they pertain to the MTEs in my study. Following this, I will identify contributions of the framework I developed and explain how it can be used to study MTEs and as a tool for professional development of MTEs. Finally, I will state the limitations of this research and the implications for future study of teacher educators.

Knowledge Base of Mathematics Teacher Educators

My study contributes to the research on the knowledge base related to the practices of MTEs by presenting an evidence-based understanding of the knowledge used by MTEs as they work to develop pre-service mathematics teachers’ knowledge for teaching mathematics. Through the analysis of the data collected from four MTEs over the course of one semester, in which they taught a mathematics content course for pre-service elementary teachers, my study focuses on the following research questions:

1) What goals for teaching and learning do MTEs develop by teaching of content courses for pre-service elementary education students? How do these goals draw on particular forms of knowledge and inform classroom practices?

2) What practices do MTEs foster by the teaching of content courses for pre-service elementary education students? How are these practices related to their goals and knowledge?
3) Why do MTEs draw on particular knowledge in their teaching of content courses for pre-service elementary education students? How are these forms of knowledge related to their goals and practices?

Though not generalizable to all MTEs, I believe the findings from this study of four MTEs will be interesting for the teacher education community. As a result of my research I learned that although developing mathematical content knowledge was a prominent goal for my participants, developing positive dispositions towards mathematics and mathematics teaching was considered an equally important course outcome. I learned that the MTEs I studied felt poorly prepared for the work of teacher educating and drew from their knowledge-in-practice to improve their knowledge and practice. Finally, I learned how some MTEs attend to issues of equity in their mathematic content courses, something I wondered about while designing this study. I will discuss each of these findings in more detail, below.

One of the prominent teaching goals that every MTE in my study had was to develop PSTs’ confidence in their mathematics and teaching abilities. Although it is important for mathematics teachers to develop confidence in their students, it is crucial that this is done in mathematics content courses for PSTs. Elementary PSTs, especially, often feel that they are not good at mathematics and have anxiety around taking mathematics courses (Ball, 2009). One of the ways that this work of MTEs is different than the work of mathematics teachers is that pre-service elementary teachers sometimes come to teacher education with a false sense of confidence in their mathematics abilities (CBMS, 2001; 2012) stemming from their procedural education. Others, still stemming from their procedural education, come to teacher preparation with misconceptions about mathematics teaching and learning. Whereas elementary school students often lack confidence due to a weak foundation, or faulty past mathematics teaching, PSTs often have a foundation that is full of holes and misconceptions. In both cases, however, deep conceptual understanding is the key to developing confidence and conceptual adaptability.

Mathematics teachers need to develop conceptual understanding and procedural fluency, as well as confidence in mathematics abilities, in their students. By studying the practice of four MTEs, I learned about the importance of confidence building in a mathematics content course for PSTs. To be fair, I knew,
when I started my research that there is more to teaching PSTs than building their content and pedagogical knowledge. One of the reasons I chose to study instructors of elementary PSTs, as I described in my introduction, was because I had an inkling that mathematics content teacher educators had a special charge due to the interdisciplinary nature of elementary education. Elementary teachers need to know how to teach a variety of disciplines to a broad range of grade levels, as opposed to secondary teachers who need to know how to teach one discipline to four grade levels. That special charge turned out to be the development of PSTs’ confidence in their mathematical abilities and their abilities to teach mathematics.

Tina considered building confidence as the ultimate goal of the course and brought it up before bringing up the goal of mathematical content knowledge development. Alec communicated his goal of developing confident students in every aspect of his course, including explaining their grades in terms of PSTs’ confidence to do mathematics. When I asked about his course goals, Sal talked at length about helping students develop deep knowledge of mathematics, but added at the end, “you know, so it sounds silly, but I want folks to like math. I think that this course is an opportunity for them to find something to like about mathematics even if they didn’t before” (initial interview, January 28, 2015). Fay considered students’ capacity to no longer fear mathematics as one of the most vital outcomes of her teaching strategies. When I asked her how she engaged students with mathematics, she told me about a game she always plays with her students in the first day of class, called “Two Truths and a Lie.” Everyone in the class writes down two true statements about themselves and one that is not true. One of her truths is always the fact that she failed mathematics in high school. She said the purpose of this activity is to show students that they can learn to like, appreciate, and understand mathematics regardless of their past experiences with it.

I expected to find that all four of the MTEs would see students’ development of mathematical content knowledge as their primary goal. But it turned out that although this goal was set as a high priority, MTEs went beyond the goal of just knowing to include knowing mathematics and being confident in their ability to teach it to others as equally important goals.
In my study, knowledge was examined through the constructs of knowledge-for-practice, knowledge-in-practice, and knowledge-of-practice (Cochran-Smith & Lytle, 1999). I learned that most of the goals and practices of MTEs in my study were influenced by their knowledge-in-practice—practical knowledge resulting from teaching and reflection on teaching.

Everyone except for Fay, who had a Ph.D. in Mathematics Education, had little formal training and skill-building for the job of teacher educating, thus they constructed this knowledge as they taught and made decisions about future instruction based upon their reflection on the teaching. Fay was the only one who stated that her doctoral program focused on developing her knowledge and abilities to educate teachers. Although, she also stated that the program was focused on developing schoolteachers into teacher educators, and she, having only experienced teaching college-level students, felt that some of the courses did not reach her as a learner. Fay holds a Master’s and a Ph.D. in Mathematics Education and a Bachelor’s in Mathematics. Her dissertation research focused on using Ball’s, et al. (2008) mathematical knowledge for teaching framework to study the knowledge development of elementary education teacher candidates. She attributed this research, not her coursework, as being responsible for much of her knowledge and confidence in regard to educating PSTs.

Alec holds a Master’s and a Bachelor’s in Mathematics and is currently perusing a doctorate in Mathematics Education, but he explained that this degree does not address teacher educating. Before becoming a teacher educator, he taught college-level mathematics courses to non-education majors and volunteer-taught mathematics at an elementary school for two years as part of an alternative certification program. Alec was very cognizant of how heavily he drew from his knowledge-in-practice. He acknowledged that the knowledge for teaching mathematics for teachers is different from the knowledge for teaching mathematics and explained that he developed most of this knowledge through teaching the course.

Tina holds a Bachelor’s, a Master’s, and Ph.D. in Education, all with a focus on mathematics. Before becoming a teacher educator, she taught high school mathematics for 20 years and college mathematics for one year. Her sentiment in regard to the teaching of the courses as the contributing factor to her
knowledge base for teacher educating was very similar to Alec’s. She has taught the mathematic content course multiple times and stated that she learned much about teaching by going through the experience and learning from her students. In addition, she drew heavily on her knowledge of the teaching competencies and the multi-step learning process, both developed by educators that took part in the national grassroots-based mathematics intervention project she directed. While reflecting on her knowledge and experiences, Tina acknowledged that she needed to enhance her knowledge base to include understanding that was not developed by a university program or a professional development.

Sal holds a Bachelor’s, a Master’s, and a Ph.D. in Mathematics. He has taught courses for a mathematics endorsement to in-service teachers but has no experience teaching at the school level. In addition to teaching mathematics content courses to PSTs, Sal still teaches mathematics courses to undergraduate students not pursuing teaching degrees. As contributing factors to his knowledge base for teacher educating, Sal credited his opportunities to teach mathematics to undergraduates and mathematics for teaching to in-service teachers.

In addition to knowing all the different ways students can misunderstand mathematics, it is also important to consider the different understandings students bring to a teacher education program. Whether influenced by their cultural, racial, linguistic, educational, or work experiences, students who are learning mathematics for teaching already have a diverse understanding of mathematics. While designing this study, I was curious to find out how much attention MTEs give to issues of class, race, culture, language, gender, and other identities in their mathematics content courses. Drawing from experiences taking these courses myself, and sitting in as an observer in others, I was anticipating issues of equity would not come up in these courses. But, in fact, half of the MTEs in my study addressed these issues explicitly. The other half attended to the first three out of the four features of democratic education: (a) a problem-solving approach to teaching mathematics, (b) students being taught in a way that that provides numerous opportunities to access and process mathematical ideas, (c) students communicating mathematical ideas with care and respect, and (d) students being encouraged to critically evaluate mathematical data for social and personal action (Ellis & Malloy, 2007).
Multiple data fragments, from interviews, classroom observations, and handouts, converged to confirm that Alec considered issues of equity while teaching mathematics content courses to elementary PSTs. He did this by addressing the need to attend to the individual nature of students’ learning no matter how different or non-traditional their methods may have to become in order to reach these students. Alec felt that it is a teacher’s job to validate students’ correct answers no matter how non-traditional. This was the reason why Alec worked so hard to get the PSTs to understand a variety of methods, models, and representations. Informed by the literature on mathematics teaching and learning, Alec was aware of how easy it is to discourage students in mathematics and make them feel that their contributions are not as good as that of the teacher or other students. Because of this, Alec developed activities, such as the video series I described in his case, which asked PSTs to look at the different ways that students may potentially solve problems and show interest in these ways, rather than admonish them. Alec told me that he knew that he could not reach every student and that some of the students will continue to use the standard algorithm as the only correct way. However by exposing PSTs to the possibility of other ways, he was planting the seed that would allow them to critically look at the way mathematics is taught and be more informed when making decisions about what kind of mathematics teaching strategies to employ.

Tina also made PSTs aware of her intentions to develop their understating and appreciation of non-traditional methods and ways of thinking, in her syllabus she states “Prospective teachers should investigate the importance of providing mathematical experiences for students who have been denied access in anyway to opportunities in mathematics as well as those who have not” (Mathematics for Elementary Teachers 2 Syllabus, Spring 2015).

Tina believes that effective teachers and MTEs should have accurate listening, cultural understanding, and intellectual flexibility—the three competencies she helped conceptualize while working with other mathematicians as part of a national grassroots-based mathematics intervention project. Accurate listening is the ability to actively listen in order to understand the thinking, feeling, and behavior of students in order to respond in a manner that meets their needs. Cultural understanding speaks to the ability to make an effort to understand and appreciate cultures that are different from your own and to challenge others to
be open-minded when looking at the world. Conceptual adaptability is the ability to connect mathematical topics to each other and other topics and experiences in a way that is novel and creates new and deeper understanding.

In her own practice, Tina was conscious that many of her students were first-generation college students, like her, and she did everything in her power to make sure that they finished their degrees. She was hopeful, that like her they would become teachers in inner-city schools that are often underfunded and drive many effective teachers away.

Above, I have presented several examples of how with the guidance of my research questions and the goals-knowledge-practice framework, I was able to illuminate important aspects of the work of teacher educating. The goal of the framework is to offer a new lens through which to study MTEs' practice by understanding and analyzing their goals, knowledge, and ideologies. To borrow from Schoenfeld (2011) "the better we can understand a range of complex knowledge-intensive activities, including teaching, the better we can help people become effective at them" (p. 3). Therefore, by offering this lens for the study of MTEs' practice, I hope to become more effective as a mathematics teacher educator and help others do the same.

**Framework for Professional Development of Teacher Educators**

Dengerink, Lunenberg, and Kools (2015) define professional development as “an internal process in which professionals engage within a formal or informal framework. The process is rooted in critical self-analysis of professional practice” (p. 80). Throughout the study, participants voiced their appreciation of the reflective element that being a part of my research has brought into their practice. They stated that thinking about interview questions in advance made them more purposeful in their goal setting and instruction and that having the conversations about the lesson after they delivered it generated new goals and knowledge, which influenced future instruction.

Tina, an experienced teacher educator, who has 43 years of teaching experience (17 at the high school level and 26 at the university level) said, referring to the structure of my pre- and post-observation interviews:
It helps me to think about how I’m engaging students, what I’m using, what my anticipated challenges might be because although it’s in the back of my mind, until you gave me those questions and I actually sat down and I had to communicate that to you . . . . Now, on my way to teach, I think about that. I have some kind of rationale for whatever we do, but it’s not as clear as it was before, which means that I need to revisit all that (post-observation interview, May 21st, 2015).

In a sense, being a participant in my dissertation study spawned knowledge-of-practice by making their practice a site of inquiry.

Alec had a goal of students being independent learners and used his knowledge of Cognitively Guided Instruction (CGI) and other student-centered resources to make this goal a reality. However, a look at his practice revealed that his instructional moves and activity structures created a classroom where direct instruction was the primary tool of learning. By examining his goals and knowledge, he was able to see this disconnect and address it for future lessons. Another goal of his was to create a community in the classroom by having students work in different groups and getting to know one another. In one of the post-observation interviews he stated this goal to me and talked about the ideal physical classroom set-up. Based on his own reflection after our conversation, he generated a plan to creatively get students into new groups by having them line up in order of their birthdays and then forming a group with the students next to them. This move shows the power of reflection-on-action (Schön, 1987), especially reflection in a setting where educators co-join their efforts to construct new knowledge. This type of knowledge is inherent to the notion of knowledge-of-practice. “Through inquiry, teachers across the professional life span—from very new to very experienced—make problematic their own knowledge and practice, as well as the knowledge and practice of others, and thus stand in a different relationship to knowledge” (Cochran-Smith & Lytle, 1999, p. 273). Although not intentional, being a participant made him take a critical look at the relationship between his goals, knowledge, and practice and make necessary adjustments.

Although my interview protocol was designed to answer my research questions about my participants’ practice, not surprisingly the participants started to think about their answers in advance and sometimes made decisions related to the lessons based on their answers. For example, if before participating in my study, the instructors did not consider the challenges that might arise for them or their
students, after hearing me ask about anticipated challenges week after week, they inevitably began to think about challenges in advance. The same goes for assessment. In the pre-observation interviews I asked about success criteria and factors that would make the lesson less than successful. At first, the MTEs took some time to consider the answer to my inquiries. But after a few weeks, I could tell, by the increased speed with which they were able to identify success criteria and factors that would make the lesson less than successful, they had thought about these answers beforehand, while planning the lesson, or, in Tina’s case, on the drive over to teach the course.

In addition to offering a lens for the study of mathematics teacher educators, I believe my framework can be used as a tool for professional development of MTEs. In using the goals-knowledge-practice framework, teacher educators can work together to examine their goals for the current course they are teaching alongside their broader goals for teacher education and their ideologies. By drawing on the knowledge they have developed and considering how their knowledge needs to evolve in order to meet the goals they see as not being met, they would engage in a deep examination of their practice. This professional development would be driven, implemented, and assessed by the teacher educators themselves. New knowledge and insights would be constructed as teacher educators discuss the video of their practice, alongside their stated goals and ideologies, in collaboration with other teacher educators. The variety of goals, knowledge, and ideologies that a learning community consists of promotes the stretching of one’s goals and the evolution of one’s knowledge.

My intention is to provide an analytic framework for theorizing teacher educator learning through the analysis of their goals, ideologies, knowledge, and practice. Although the work of analyzing one’s teaching can be achieved in solitude with the use of self-study, a great depth of analysis and reflection can be accomplished if teacher educators examine their work within communities of inquiries, as discussed in Cochran-Smith and Lytle (1991, 2009). Analyzing one’s practice is an enormous task, but the framework provides a structure for making the task less daunting by focusing on certain aspects of one’s work and the relationships between these aspects.
We know that experience plays a large role in what teachers do and how they do it, but we also know that learning experiences can shift goals and ideologies. Alec is an example of someone who started out with the ideology of an old humanist (Ernest, 1991), valuing pure mathematics and being hesitant of creating development classes for teachers, because of his views that everyone should go through the same program. But as his knowledge base developed, through professional development, reading, and his work as an MTE, his ideology shifted to that of a public educator (Ernest, 1991), who believes that teachers do need additional experiences that expose them to alternative viewpoints and get them out of their comfort zone.

Cochran-Smith and Lytle (1999, 2009) describe a professional development model called, inquiry as a stance. Describing the way one engages with inquiry, Cochran-Smith and Lytle (2009) write, that inquiry is:

A worldview and a habit of mind—a way of knowing and being in the world of educational practice that carries across educational contexts and various points in one’s professional career and that links individuals to larger groups and social movements intended to challenge the inequities perpetuated by the educational status quo. (p. vii)

In conjunction with the goals-knowledge-practice framework, as a model for professional development of teacher educators. Goos (2009) writes,

Many mathematics education researchers have begun to draw on sociocultural theories in proposing that teachers’ learning is better understood as increasing participation in socially organised practices that develop their professional identities (Lerman, 2001). Such sociocultural approaches to mathematics teaching and learning involve “frameworks which build on the notion that the individual’s cognition originates in social interactions . . . and therefore the role of culture, motives, values, and social and discursive practices are central, not secondary” (Lerman, 1996, p. 4) (p. 2010).

This statement is consistent with many of the principles of inquiry as a stance, which is based on the premise that knowledge is socially constructed by teachers who work together, all participants are fellow learners, and inquiry is looked at as a larger effort to transform teaching and learning (Cochran-Smith & Lytle, 1999; 2009). One of the most important questions about teaching that Shulman asked in 1986, and many have continued to ask since, is: how is subject matter transformed from the knowledge of the teacher into the content of instruction? The framework of goals, knowledge, and practice can begin to answer that question.
I propose that professional development of teacher educators begin with identification and critique of goals, knowledge, ideologies, and practice, within a supportive learning community of other teacher educators. By engaging in peer review of teaching artifacts, such as recordings of classes taught, syllabi, or student work, teacher educators reflect on the numerous aspects of their practice, get feedback from critical friends, and revise to achieve better alignment between goals, knowledge, ideologies, and practice. Thus, although the inquiry begins with knowledge-of-practice, later, stimulated and inspired by these inquiry groups new knowledge-in-practice may develop, as teachers implement some of the revisions they brainstormed about in the inquiry groups. The last step, knowledge-for-practice, a targeted look into the literature and other resources for educators, is guided by the outcomes of individual and group inquiry.

Finally, since the methods of this study significantly benefited my own development as an MTE, I will describe my methods in a way that can be used for others considering similar work of developing their own practice through a study of others.

**Contribution to my knowledge base**

Unlike many practicing teacher educators, who did not set out to become teacher educators, (Goodwin et al., 2014; Masingila et al., 2012; Zeichner, 2005), I knew I wanted to become a teacher educator from the moment I sat in my first teacher education course. When considering doctoral programs, I considered the impact of this degree on my work as a teacher educator. My search for an institution that had a program which focused on preparing teacher educators, fell short, not surprisingly, since “a quick sweep through doctoral programs in the U.S. reveals only a handful institutions offer educator preparation, even while over 1400 institutions are in the business of preparing teachers—and therefore in the market for hiring teacher educators” (Goodwin & Chen, 2016, p. 160, emphasis in the original). Instead I chose an institution that had, in my opinion, an interesting program of study and body of instructors and was conveniently located. While working on my doctorate degree I took courses on learning mathematics, teaching mathematics, mathematical identity, and mathematics for social justice, as well as courses that taught me about understanding and conducting research. However, developing a knowledge base for teacher educating was not one of the intended learning outcomes of the program.
Therefore, I learned little about the knowledge and skills needed to become an effective teacher educator, nor did I instruct teacher education courses, or conduct research in the area of teacher education, before starting my dissertation work. While many may consider my K–12 teaching experience, which included working with student and beginning teachers to help them develop planning, instruction, and assessment skills and strategies, to be sufficient preparation for the work of teacher educating (Berry, 2007; Dinkelman at al., 2006; Lunenberg & Hamilton, 2008; Zeichner, 2005), I knew that it was not. Given a lack of relevant experience in the area of university-based teacher education and a deep understanding of the knowledge of second-order work and higher education, it should not be surprising then that my dissertation research focus is the understanding and examination of the second-order work of teacher educators (Murray and Male, 2005).

Zeichner (2005) states that novice teacher educators need to immerse themselves in the study of the literature in the field of teacher education and self-study of their work in teacher education. Although some doctoral candidates take the opportunity to instruct in teacher education programs, I did not do so, resulting in a rather limited understanding of the teacher education space as it pertains to university-based teacher preparation programs. However, by conducting this research study I was able to engage with teacher education literature and examine the work of teacher educators. Through conducting the literature review for my study, I familiarized myself with a range of perspectives and research on teacher preparation and knowledge and skills for teaching mathematics. Through observing and interviewing practicing teacher educators, I was able to investigate the application of this research and gain a deeper understanding of the knowledge, practice, and goals of teacher educators. I feel that because I was not the instructor of the courses I studied, I was able to gain an understanding of their interworking without worrying about the implications of my study on my instruction.

The different components of my dissertation work, such as the literature review, data collection, and data analysis, provided the learning experiences I felt were necessary in order to become a university-based teacher educator. These experiences were so beneficial that I would like to highlight a few of my methods and procedures in order to make them available for those who are becoming teacher educators,
or for departments of education who want to provide their students with some type of field experiences. I am not suggesting that every prospective teacher educator conduct dissertation research of other teacher educators, although I agree with scholars who believe that one of the goals for doctoral dissertations for teacher educators should be to bolster up the knowledge base for teacher education (Nof et al., 1999).

Methods for Studying MTEs

The first task in research that focuses on teacher educators is to come up with focused questions of study, since the understudied area allows for many entry points. In my research I focused on the relationship of goals, knowledge, and practice. The second step is to develop interview questions and observation foci, which align with the research questions. To this point my interview questions aligned with my intentions to study and examine the goals, knowledge, and practice of teacher educators, as shown below in Table VIII. Complete interview protocols are included in Appendices A – C.

Aligning my interview questions with my research questions reminded me of the goals of my research and provided a frame of mind for observations and looking at the handouts collected during class sessions. I feel that this focus made it possible for me to collect a vast but manageable amount of data about my participants.

Another important part of my data collection was the use of memos as a preparation tool for analysis. After completing each class observation, and the accompanying pre- and post-observation interviews, I listened to the interviews and went over my field notes and handouts collected. During this time, I wrote memos to document my observations about what I saw and heard and brainstorm potential codes. This process of revisiting my data shortly after it was collected and writing operational and theoretical memos provided a way to make sure that the data I collected addressed the goals of my dissertation research. It also allowed me look up relevant literature that came up during the observations, prepare questions that were not answered in the interviews and observations, and get a head start on analysis.
<table>
<thead>
<tr>
<th><strong>Goals</strong></th>
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<tbody>
<tr>
<td>• What do you see as the goal(s) of teacher preparation, general, and mathematics?</td>
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<tr>
<td>• What is the purpose of the course that you teach towards this goal(s)?</td>
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<tr>
<td>• What are your teaching goals for this lesson? Why these goals?</td>
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<tr>
<td>• What are your learning goals for this lesson? What are the main mathematical ideas that the students should get from this lesson?</td>
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<tr>
<td>• Did you accomplish all the goals/complete all the activities listed on the syllabus? Why or why not?</td>
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<tr>
<td><strong>Knowledge</strong></td>
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<tr>
<td>• When planning for mathematics content courses for pre-service teachers what type of theories do you draw on?</td>
<td></td>
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<tr>
<td>• What resources do you use to plan for this math content course? Why?</td>
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<tr>
<td>• What knowledge do you draw on while teaching the course/interacting with pre-service teachers?</td>
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<tr>
<td>• What knowledge are you drawing on to help students reach the goals of this lesson?</td>
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<tr>
<td>• What knowledge are you drawing on to help students reach the goals of this lesson?</td>
<td></td>
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<tr>
<td>• What sources did you draw on to prepare for this lesson?</td>
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<tr>
<td>• Which instructional materials and practices do you plan to use during this lesson and why?</td>
<td></td>
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<tr>
<td>• Does working with elementary vs. secondary pre-service teachers influence the knowledge you draw on? Why or why not?</td>
<td></td>
</tr>
<tr>
<td>• How is the professional knowledge of a teacher educator different from the professional knowledge of a K–12 teacher?</td>
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<tr>
<td><strong>Practice</strong></td>
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<tr>
<td>• What type of courses for pre-service teachers have you taught?</td>
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<td>• Have you taught this particular course that I will be observing before? How many times?</td>
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<tr>
<td>• What is the organization of your class sessions?</td>
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<tr>
<td>• What do you anticipate will be the most challenging teaching issues in this lesson? How will you address these challenges?</td>
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<tr>
<td>• What do you anticipate will be most the most challenging learning issues in this lesson? How will you address these challenges?</td>
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<tr>
<td>• If you had to teach the course again, would you make any changes to the organization of the class sessions? Why or why not?</td>
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<tr>
<td>• If you had to teach the course again, would you make any changes to the organization of the course/syllabus? Why or why not?</td>
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<td>• Do you plan to use the same instructional materials if you teach this course again? Why or why not?</td>
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<tr>
<td>• Do you plan to use the same instructional strategies if you teach this course again? Why or why not?</td>
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To analyze my data, I used a process of examining and re-examining the data in increasingly refined ways in order to generate a rich interpretation called thematic analysis. This process involved six phases: “familiarizing yourself with your data, generating initial codes, searching for themes, reviewing themes, defining and naming themes, and producing the report” (Braun & Clarke, 2006, p. 87). As already stated, I began the first stage, familiarizing myself with the data by revisiting my interview recordings and field notes and writing memos on the data before all of the data was collected. Once all of the data was collected, I transcribed it, during which time I wrote more memos about codes, definitions, questions, and initial analysis.

During the second phase, generating initial codes, I used open coding in the form descriptive and in-vivo codes (Miles, Huberman, & Saldaña, 2014) to generate over two hundred codes. To search for codes (phase three), I used the method of writing each code on a piece of paper and sorting them into groups to organize the codes and generate potential themes. This was done in multiples stages. I first grouped all the codes into a multitude of categories based on the different stages of teaching (i.e. pre, post, and during), and different processes (i.e. goal setting, instructional moves, identifying and addressing challenges, assessment, planning, etc.). Then, I revisited my research questions, grouped the codes into the categories of goals, knowledge, practice, and other. Once the broad groups of codes were created, I categorized the codes within each group further, often drawing on the literature that I had read about goals, knowledge, and practice of educators. This allowed me to move on to phase four, reviewing the themes. To do this, I revisited my field notes, interview transcripts, and documents to make sure that the themes were evident across the three types of data. To complete phase five, defining and naming the themes, I went back to my research questions and relevant literature. At this stage, I drew heavily on the work of Cochran-Smith and Lytle (1999) and Scoenfeld (2011) to define the major and minor themes and organize them into a framework. Finally, I wrote up the report (phase six) in the form of individual case studies and a cross-case analysis.
In addition to contributing methods for how to study MTEs to the research on teacher educators, my study contributes a framework that highlights some of the components of the complex knowledge base of the four MTEs I studied. These findings are described below.

**Reflections on my research**

As a result of personal experiences and my engagement with the research literature, I became interested in exploring the knowledge base of MTEs and the different ways that knowledge is conceptualized in and for practice. To do this, I drew from a number of different scholarly perspectives on knowledge. Some of these conceptualizations included knowledge-for-practice, knowledge-in-practice, knowledge-of-practice (Cochran-Smith & Lytle, 1999), and knowledge of human resources (Chauvot, 2009). One could read into my dissertation that I was juxtaposing differing and diverging conceptualizations, each with its own set of considerations and implications. For example, knowledge-for-practice is developed through one’s educational experiences and conjures up assumptions about who produces and who consumes this knowledge—researchers produce knowledge for practitioners to use. There are also assumptions about having more education or coursework as related to the quality of one’s practice as an instructor.

The depiction of my framework also could be misread to suggest that I am claiming all forms of knowledge are equally relevant to my analysis or to the individual MTEs in my study. However, there is data in the study to suggest otherwise. In the analysis of the different data sources, for example, knowledge-in-practice surfaced as the knowledge that the MTEs draw on most prominently to make decisions for planning and delivery of content in mathematics content courses for teachers. This is perhaps not surprising given that one of the aims of my research study was to understand MTEs’ knowledge as it manifested itself in practice. Consequently, a large amount of the data came from classroom observations and reflections on observations. This is not to say that their knowledge-of-and -for-practice were not important to the MTEs or that they did not inform their goals and practices. There is documentation in the cross-case analysis and the analysis of the individual cases that show how participants utilized their education (knowledge-for-practice), and their research and participation in
inquiry groups (knowledge-of-practice) in preparing for, teaching, and reflecting on the content courses. Different conceptualizations of knowledge are helpful in understanding participants’ goals, practices, and ideologies, but as I continue with this research I now realize it may be useful to hone in on a specific conceptualization.

In addition to the prominence of knowledge-in-practice as a source of knowledge for the MTEs, other aspects of their work that I did not account for when designing the study or collecting the data, manifested during analysis. Identities, ideologies, and beliefs of the MTEs had a strong impact on their goal setting and knowledge development. I did my best to account for these by deducing participants’ deeply-held beliefs and ideologies, and conceptualizing them in terms of knowledge. I now wonder if the case descriptions would be even more rich if I had accounted for constructs such as identity or ideology at the start of my research. For example, at the initial interview, Alec stated that he is well-read in terms of mathematics for social justice practices and would like to make them a priority in his teaching of mathematics content courses, but he lacked the time to plan and deliver instruction that engaged PSTs with that concept. As I analyzed his data, however, I found that many of his practices were consistent with what Ernest (1999) calls a public educator. By presenting students with multiple representations of the same problem and encouraging them to approach problems in a variety of ways, Alec was ensuring that students develop sense-making and critical thinking skills to analyze mathematics and see themselves as mathematics doers. This philosophy of expanding and challenging students’ ideas and beliefs influenced Alec’s goals and practices. Knowing this, when I extend this research I will consider the role of identity, ideologies, and philosophies at the start of my research.

In explaining my analysis, I stated that the framework was both a priori and emergent. Although I did not consider goals and ideologies in my initial guiding questions, my interview protocols focused on goals and included questions that asked participants to think about why they were drawing on particular forms of knowledge and resources, which often brought out their deeply-held beliefs and ideologies. That being said, the questions did not always reveal answers I could have predicted. For example, when I asked participants about the effects of where they teach on how they teach, several were not able articulate
answers that explained their goals and practice. Some talked about university values but not their own, and others admitted that they had not thought about the effect of their institutional contexts on their teaching practice.

Other tension and contradictions result from the representations of my framework. As I noted earlier, knowledge can be conceptualized in different ways, but so can goals and practice. I started with the categories of goals, knowledge, and practice. Analysis showed that these categories are collaborative, thus I added the two-way arrows without specification of what the arrows mean, knowing that part of the meaning is embedded in individual cases. I used this emergent framework to analyze the data and applied it to the case analysis, the individual cases, and sometimes individual episodes. The framework with all of it’s components is already complex, but as I reflect on its use, I know that there is even more complexity to be explored. The way the different components interacted with each other is different for every MTE, class session, activity, and interaction. The arrows themselves, for example, may not go both ways for each participant or classroom episode. Meaning that sometimes goals inform practice, but practice does not inform goals. Sometimes knowledge development influences goals but not practice, and so on. If I were to apply the framework to segments of the class sessions, more meaning would begin to develop, and the relationships between the different pieces might reveal even more complexity.

Limitations

Because I recruited participants who self-identify as teacher educators, the MTEs that agreed to be in my study were all experienced teacher educators. Thus, I was not able to collect any data about the goals, knowledge, and practice of novice teacher educators. Additionally, all of my participants were located in the same metropolitan area, possibly limiting perspectives due to a narrow geographic area. Finally, I chose to focus on a particular course, mathematics content for pre-service elementary teachers, which limited the types of MTEs recruited, since those courses are generally taught in the mathematics department, as opposed to mathematics methods courses, which are taught in the education department.

Another limitation of my study was the lack of data from the pre-service teachers. This data was purposefully not collected, since they were not the focus of the study. However, data from student work,
student-MTE interactions, and student interviews would have added another level to the study. Additionally, I did not collect any data from university staff and administrators.

Teaching is a complex activity that includes many components. Since I could not cover all of the aspects of teaching, I chose to concentrate on certain goals (teaching, learning, and curriculum), certain ways of looking at knowledge (knowledge-of, -in, and -for-practice), and certain practices (instructional moves, activity structures, and decision-making). Certainly, because of the focus on these components of the work of MTEs, many others were left out and should be considered in future research studies.

Lastly, ideologies emerged as an important consideration in relation to knowledge, practice, and goals. My decision in this study was to include ideologies as a component of knowledge. While this was consistent with my data and analysis, it is also possible to include ideologies as a fourth-component of the framework. Additional research would be helpful in examining the ideologies of MTEs.

Despite these limitations, I feel that I drew important conclusions from the data I collected, given my sample size and type, and the timeframe of the study. The results, although not generalizable, provide building blocks for future study and professional development of teacher educators from multiple sites, disciplines, and levels.

**Future research**

My research focused on four university-based experienced MTEs from the same metropolitan area, as they taught mathematics content courses to pre-service elementary teachers. Based on this description of my participants, future research could expand to include a more diverse body of participants. For example, future participants could include university- and school-based teacher educators; new and experienced teacher educators; teacher educators across different disciplines; teacher educators from different metropolitan areas; and teacher educators who teach different courses within a teacher preparation program.

For example, a future research study could focus on the work of university- and school-based teacher educators in order to understand the differences and similarities in the goals, knowledge, and practice of the two types of teacher educators. Van Zoest, Moore, and Stockero (2006) wrote about the collective
experiences of aspiring teacher educators and their mentor, an experienced teacher educator, as they collaboratively taught a course for PSTs. A variation of this study could include research about the goals, knowledge, and practice of educators of mathematics teacher educators as they prepared future teacher educators. The goals, knowledge, and practice of the aspiring teacher educators could be examined during their times as becoming, novice, and experienced teacher educators and compared to the goals, knowledge, and practice of their educator or educators.

Additionally, the focus of the study could be broadened from teacher educators to teacher educators and their students, PSTs, in order to understand the impact of an instructor’s goals, knowledge, and practice components on the knowledge and practice of their students. This study could also be extended to study and analyze the progression of their goals, knowledge, and practice as they moved from pre-service to in-service teaching and as they take on student teachers, thus becoming teacher educators themselves.

Another future direction for my research could be to use the framework of goals, knowledge, and practice to design professional development programs which will drive MTEs to take a critical look at their practice in order to see if there is a disconnect between their goals, knowledge, and practice. Especially, I believe this kind of work of examining the practice of MTEs is important for aspiring teacher educators, as research shows that they often come to the work of teacher educating lacking the knowledge and skills necessary to do this job well (Goodwin et al., 2014; Masingila et al., 2012; Murray & Male, 2005; Zeichner, 2005).

I believe it would be interesting to extend the study of the professional development of MTEs to teacher educators from other disciplines. Together, these teacher educators would critically examine their own and each other’s goals, knowledge, and practice in order to learn from and about the work of teacher educators from disciplines other than their own. This type of professional development can bring cohesion and a deeper understanding and development of shared visions and missions across teacher education programs.

Finally, as stated in the reflection I can extend the research by honing in on a specific conceptualization knowledge, such as knowledge-in-practice (Cochran Smith & Lytle, 1999) and using it
to explain goals and practices. This study would analyze MTEs’ knowledge-in-practice through their goals for practice and the practices themselves for the purpose of learning to what extent these goals can be explained by their knowledge and ideologies. This way of honing in on one conceptualization of practice would allow for in-depth analysis across cases, within each class, and within class sessions.

Conclusion

Because of the small participant sample, the results of my study are not meant to be generalized. However, I do provide important findings in the form of in-depth analysis, and contribute a framework that can help guide future study, as well as professional development of teacher educators across disciplines. The framework I propose draws heavily from the work on Cochran-Smith and Lytle (1999) on the relationship between knowledge and practice, and the work of Schoenfeld (2000, 2011) and his Teacher Model Group on understanding teachers’ decision-making practices through a framework of resources, orientations, and goals. Schoenfeld and his team used the framework to understand the decisions that teachers make, stating,

What a teacher decides to do while engaged in teaching is a function of the teacher’s goals (some of which are determined prior to the instruction and some of which emerge as the lesson unfolds), beliefs (which serve to re-prioritize goals as some goals are satisfied or new goals emerge), and knowledge (including various routines the teacher has for achieving various goals) (2008, p. 49).

The categories of my framework are not new. I have constructed the framework by building on existing literature. However, the conceptualization of knowledge as for-, in-, and of-practice, the focus on ideologies, and the focus on instructional moves and activity structures, do constitute a new lens for studying MTEs. Additionally, it is my hope that the suggestion to use the framework for professional development of MTEs will bring much needed attention to the research and practice of MTEs learning in communities. Finally, the detailed description of my research methods provides a foundation upon which aspiring teacher educators can build in order to engage in similar work for the purposes of contributing to the research community and their own professional development.
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APPENDIX A. INITIAL INTERVIEW PROTOCOL

1) Please tell me about your own educational experience – Where did you go to college and how did your education influence who you are as a mathematics teacher educator?

2) Please tell me about your professional experiences – Where have you worked/taught and how did these experiences influence who you are as a mathematics teacher educator?

3) Do you consider yourself to be primarily a teacher educator who teaches math content or a math content specialist who works with prospective teachers? Is there some other way that you characterize your role and the work that you do?

4) What type of [math content and other] courses for pre-service teachers have you taught?

5) Have you taught this particular course that I will be observing before? How many times? How was it that you first came to be involved with this course? How does this course align with your background and training or other professional experience? What are the goals you try to achieve when you teach this course? Why these particular goals?

6) Are there any general principles or assumptions (about teaching, learning, mathematics, students, education, etc.) that serve as a foundation for your planning for and practice in this course?

7) What is the organization of your class sessions? (How do you organize and structure your classes to maximize learning? Maximize engagement? Maximize the development of your students?)

8) What is the text (s) that you will use for this course, if any? Did you choose this text? Are there features of the text that you find especially appealing? Why? Are there aspects that you would change? Why?

9) Have you taught math methods courses? Which ones and to whom? How is this course different from a math methods course? In what ways are they similar? Do they require similar types of expertise? How do the demands on students differ or align?

10) (Written)* When planning for mathematics content courses for pre-service teachers what type of theories or perspectives do you draw on to inform your thinking about:
a. Learning mathematics
b. Teaching mathematics

Why do you draw on those particular perspectives and theories?

11) (Written) In addition to teaching and learning, what other considerations do you make in this course? What perspectives and theories do you draw on to address these additional considerations?

12) (Written) How familiar are you with various mathematics education reforms that have come and gone over the last two decades? How have those reforms affected you directly? Have you changed your teaching or approach to teaching as a result? Have these reforms hindered or enhanced your practice in any ways? What are your thoughts about the ways teachers and teacher educators have been portrayed in those reforms?

13) What other resources do you use to plan for this math content course? Why?

14) Does working with elementary vs. secondary pre-service teachers influence the knowledge you draw on? Why or why not?

15) (Written) In your opinion, what does it mean to be an effective mathematics teacher educator? What are some qualities that characterize this person and their practice?

16) (Written) What is the main job of a mathematics teacher educator?

17) (Written) Your students will go on to teach mathematics. You are working to help them become effective teachers. What are some similarities and differences between teachers of mathematics and teachers of mathematics teachers?

18) Are all of the students in your course math concentrators? If not, do you take part in the decision of which students will become math concentrators? What criteria do you use to decide which pre-service teachers should be math concentrators? Why these criteria?

19) (Written) Are there particular faculty members that you have known whose practice and impact in these courses is especially effective? What features characterize their practice and what makes them effective? Have you attempted to take up any of these practices? Which?
* Questions with the word **(Written)** next to them were sent to the participants in advance to be answered in writing.
APPENDIX B. PRE- AND POST-OBSERVATION INTERVIEW PROTOCOLS

Pre-observation interview questions
1) What are your teaching goals for this lesson? Why these goals?
2) What are your learning goals for this lesson? What are the main mathematical ideas that the
   students should get from this lesson?
3) What knowledge(s) do you use to reach your teaching and learning goals for this lesson?
4) What sources did you draw on to prepare for this lesson?
5) Which instructional materials and practices do you plan to use during this lesson and why?
6) What do you anticipate will be the most challenging teaching issues in this lesson? How will you
   address these challenges?
7) What do you anticipate will be most the most challenging learning issues in this lesson? How will
   you address these challenges?
8) What are your success criteria for this lesson, both in terms of teaching and learning? How will
   they be assessed?

Post-observation interview questions (sample – other questions may arise after the observation)
1) Let’s revisit the goals you outlined before the lesson. Do you feel the goals set for this lesson
   were met? Why or why not?
2) Do you feel that your choice of instructional materials and strategies for this lesson was
   successful? Why or why not?
3) What do you feel went particularly well in this lesson? Why do you think it went so well?
4) Tell me about how the anticipated challenges played out in the lesson. Talk a bit more about the
   teaching challenges? Talk a bit more about the learning challenges?
5) Did anything unexpected/unplanned happen during the lesson? If yes, what was it and how did
   you address it?
6) If you had to teach this lesson again, would you make changes to your plan or delivery? If yes,
   what kind of changes?
APPENDIX C. FINAL INTERVIEW PROTOCOL

(Sample. Other questions may arise after going over the individual data of each MTE)

Questions about overall practice
1) Please tell me about formal or informal teacher educator preparation or what you feel could be considered teacher educator preparation?
   • What skills were developed through this program?
   • What skills were developed in other ways? What were the ways?
2) How is your role as a teacher educator shaped by the context in which you work?
3) I would like to revisit a question from the initial interview.
   - The divide may be an artificial one, but what distinguishes the work that you and your students do in this course from the work you and they might do in a mathematics methods course. Are those differences important or consequential for the teacher and students? What are some similarities?
4) Can the same instructor be just as skilled in teaching both kinds of courses? What would help that be the case?
5) What are your current research interests and how do they inform your practice as an MTE?
6) How do you engage in professional development?
7) What type of professional development is important to you?

Questions about this semester
8) Overall, how do you feel this semester went? For you? For your students?
9) What are some things that went particularly well this semester?
10) What were some teaching challenges you encountered this semester?
11) What were some of the challenges faced by students in their learning and how did you and they respond?
12) Did you accomplish all the goal/complete all the activities listed on the syllabus? Why or why not? Did any new goals emerge or did you have to modify your goals in any way? Why was this necessary?

13) What are some of the ways that you checked the pulse of the course to see if it was going the way had hoped? Can you give an example of when you learned something from doing this?

14) If you teach the course again, would you make any changes to the activities and structure of each organization of the class sessions? Why or why not?

15) Would you change your teaching approach in any way?

16) If you teach the course again, would you make any changes to the organization of the course/syllabus? Why or why not? Are there any aspects of the “organization” that you are absolutely committed to? Why?

17) Do you plan to use the same instructional materials if you teach this course again? Why or why not? Are there any materials that you are absolutely committed to? Why?

18) Do you plan to use the same instructional strategies if you teach this course again? Why or why not? Are there any teaching strategies that you are absolutely committed to? Why?
VITAE

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with a minor in English

TEACHING EXPERIENCE

Assistant Professor of Secondary Education (2016 – present), National Louis University, Chicago, IL
- Teach graduate level secondary education courses in the Masters of Arts of Teaching Program
- Teach graduate and undergraduate level middle-level and secondary mathematics methods courses
- Design and implement face-to-face and blended learning education experiences

Mathematics Teacher (2005 - 2013), Farragut High School, Chicago, IL
- Designed and implemented materials for Algebra, Geometry and Advanced Algebra with Trigonometry courses
- Used the Sheltered Instruction Observation Protocol (SIOP) to address the needs of English Learners (ELs) in a mathematics classroom
- Employed culturally relevant teaching pedagogy to develop students’ mathematics and language skills

Bilingual Department Chair (2007 - 2013), Farragut High School, Chicago, IL
- Directed and administered assessments of students’ speaking, listening, reading, and writing levels
- Led professional development workshops on implementation of best practices for English Learners (ELs)
- Facilitated the work of the Bilingual Advisory Committee, a parent-led organization aimed at determining priorities regarding education of ELs
Mathematics teacher (1/2005-6/2005), Chicago Waldorf High School, Chicago, IL  
- Taught developmental Algebra, Geometry, and Advanced Algebra/Trigonometry  
- Designed and adapted personalized curriculum to suit the needs of students with wide-ranging ability levels and varied learning styles

RESEARCH EXPERIENCE

Graduate Research Assistant (2013 – 2015)  
National Science Foundation funded 5-year project  
*Improving Formative Assessment Practices: Using Learning Trajectories to Develop Resources that Support Teacher Instructional Practice and Student Learning in CMP2 (iFAST) Project*  
Principal Investigators: Alison Castro Superfine (UIC), Mara Martinez (UIC)

- Worked with a small team to develop and empirically test learning trajectories for linear functions and equations  
- Observed classrooms, conducted interviews, and collected student work to gather data about the use of formative assessment practices in middle school algebra classrooms  
- Analyzed qualitative and quantitative data related to formative assessment practices

Volunteer Research Assistant (08/2013 – 12/2013)  
*Learning Mathematics Needed for Teaching Project*  
Principal Investigator: Alison Castro Superfine (UIC)

- Designed lesson plans for weekly implementation by the instructors  
- Observed lessons and gave feedback to the instructors  
- Participated in weekly meetings with the instructors and other planning committee members

PROFESSIONAL DEVELOPMENT EXPERIENCE

Pre- and in-service teacher supervisor and mentor (2010 – 2013), Farragut High School, Chicago, IL

- Founded and directed an induction and mentoring program for first and second year teachers  
- Acted as a clinical instructor for student teachers in University of Chicago’s Urban Teacher Education Program (UTEPI)  
- Served as an on-site coordinator, mentored supervising teachers, and supervised student teachers for National Louis University’s Urban Scholars Teacher Education Program (U-STEP)

Assistant Director of Education (2003 – 2004), Huntington Learning Center, Park Ridge, IL

- Took part in interviewing, hiring, and training tutors for K–12 students  
- Created and monitored instructional programs for K–12 students  
- Performed student conferences and communicated with school faculty and staff to develop individualized programs for K–12 students

PUBLICATIONS

PRESENTATIONS


PROFESSIONAL ACTIVITIES

• National Louis University Teacher Preparation Advisory Council Member, (2011 – Present)
• Developed a school-improvement plan as part of a Practice-Based Inquiry (PBI) Team (2012)

INTERVIEWS


GRANTS

Teacher Incentive Grant, Oppenheimer Family Foundation (2008)
Project Title: Building Bridges to a Better Future
An applied mathematics project for all Advanced Algebra with Trigonometry courses at Farragut High School