Essential Spunnormal Surfaces via Tropical Geometry

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I dedicate this to my family, Robert and Candace Brasile; John, Joni and August Brasile; Michael and Heidi Brasile and Anna Carley; and Stephanie Brasile, who have been my support and inspiration throughout my life.
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SUMMARY

Let $M$ be a compact, orientable 3-manifold with torus boundary. An essential surface in $M$ is a nonempty, properly embedded, orientable surface $S$ such that $i_* : \pi_1(S) \to \pi_1(M)$ is injective, and no component of $S$ is boundary parallel or a 2-sphere. If an essential surface $S$ has nonempty boundary, then it must consists of pairwise disjoint, nonseparating closed curves on the boundary torus. Therefore, the components of $\partial S$ are pairwise isotopic, essential closed curves on the boundary torus of $M$. We call the unoriented isotopy class of the components of $\partial S$ the boundary slope of the essential surface $S$.

Let $M$ be a compact, orientable 3-manifold with torus boundary and $S$ be an essential surface in $M$. Take a basis $\mu, \lambda$ for $H_1(\partial M; \mathbb{Z})$. Let $\gamma$ be a component of $\partial S$, and give $\gamma$ an orientation. Then $[\gamma] = p\mu + q\lambda \in H_1(\partial M; \mathbb{Z})$ for some $p, q \in \mathbb{Z}$. Now the boundary slope of $S$ can be represented by $p/q \in \mathbb{Q} \cup \{\infty\}$. Notice that the ratio $p/q$ is consistent with both orientations of $\gamma$.

We denote the set of boundary slopes of essential surfaces in $M$ by $\text{bs}(M)$. The set $\text{bs}(M)$ is an important object. The relationship between $\text{bs}(M)$ and Dehn fillings of $M$ has produced several fundamental results (1; 2). Furthermore, $\text{bs}(M)$ gives insight into which surfaces are essential in $M$. Characterizing the essential surfaces in a 3-manifold is the first step in finding a hierarchy of the manifold, a decomposition which is the basis for many classification theorems.

A. Hatcher (3) showed that if $M$ is a compact, orientable 3-manifold with torus boundary, then the set $\text{bs}(M)$ is finite. There have also been many papers which classify this finite set
bs($M$) for certain classes of 3-manifolds with torus boundary. A. Hatcher and W. Thurston (4) gave an algorithm for finding bs($M$) when $M$ is a 2-bridge knot complement. M. Culler, W. Jaco and J. Rubinstein (5), as well as W. Floyd and A. Hatcher (6) both gave algorithms to find bs($M$) for once-punctured torus bundles. A. Hatcher and U. Oertel (7) did the same for Montesinos knot complements with N. Dunfield (8) implementing this algorithm in a computer program.

For general knot complements, W. Jaco and E. Sedgewick (9) have given an algorithm for computing bs($M$) for any knot complement $M$ using a standard triangulation of $M$. M. Culler has used the result that boundary slopes of the Newton polytope of the A-polynomial are boundary slopes in bs($M$), which was proven by D. Cooper, M. Culler, H. Gillet, D. Long and P. Shalen (10), to find boundary slopes by computing the A-polynomial for many knot complements. However, both of these processes often involve huge calculations, and there are still relatively simple knot complements where bs($M$) is unknown.

Therefore, it is still of interest to give simple criteria for a rational number to be a boundary slope of a knot complement. The main result in this thesis gives simple, sufficient conditions for a surface with nonempty boundary to be essential in a compact, orientable 3-manifold with torus boundary. These criteria are then used to list boundary slopes of essential surfaces in several knot complements. Some of these boundary slopes have not previously been calculated to the author’s knowledge.

The main result of this thesis is largely based on the ideas of N. Dunfield and S. Garoufalidis (11). Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, orientable 3-manifold with torus
boundary. It was shown by N. Dunfield and S. Garoufalidis using a result from Y. Kabaya (12), that any vertex surface in $M$ with a normal quad in each tetrahedron of $T$ is essential. These are easily checked sufficient conditions for a spunnormal surface to be essential in $M$.

In this thesis we will show any vertex surface that is contained in an isolated ray of the set of spunnormal surfaces is essential in $M$. This removes the previous requirement that there be a normal quad in every tetrahedron.

Chapter 1 gives some basic definitions.

Chapter 2 introduces ideal triangulations of compact, orientable 3-manifolds with torus boundary and cell decompositions of 3-manifolds in general.

We then develop the theory of spunnormal surfaces with respect to an ideal triangulation, which are surfaces that intersect the tetrahedra of the ideal triangulation in such a way that the surfaces can be represented combinatorially. The theory of spunnormal surfaces was introduced by W. Thurston (13) and is presented in Chapter 3.

In Chapter 4 the deformation variety is introduced. The deformation variety is a complex variety which parametrizes hyperbolic structures on the 3-manifold, and was also introduced by W. Thurston (13). There is a map from the deformation variety to the character variety of the manifold. The character variety of a 3-manifold was developed in work by M. Culler and P. Shalen (14). They used Bass-Serre theory to associate ideal points of the character variety to essential surfaces in the 3-manifold. Therefore, we are interested in finding the ideal points of the deformation variety.
Chapter 5 works towards this end by introducing concepts from tropical geometry. The Fundamental Theorem of Tropical Geometry associates vectors in a set called the tropical variety to ideal points of the deformation variety. A result of B. Osserman and S. Payne (15) is then presented which gives criteria for a vector to be in the tropical variety of the deformation variety.

The main result is then presented in Chapter 6 which shows that any spunnormal surface in an isolated ray of the set of spunnormal surfaces is essential in the 3-manifold. As previously stated, this result generalizes a result of N. Dunfield and S. Grouulfaldis (11).

The final chapter gives examples of boundary slopes of essential surfaces found using this result.
CHAPTER 1

INTRODUCTION

Definition 1.1 (Embedded manifold) Let $M$ be an orientable $m$-manifold with boundary and $N$ be an $n$-manifold with boundary where $n \leq m$. An embedding of $N$ into $M$ is a map $\varphi : N \to M$ which is a homeomorphism onto its image $\varphi(N)$ with the inherited subspace topology from $M$. We abuse notation and denote the image $\varphi(N)$ as $N \subseteq M$. We say that $N$ is embedded in $M$.

Definition 1.2 (Properly embedded manifold) Let $M$ be an orientable $m$-manifold with boundary and $N$ be an $n$-manifold with boundary where $n \leq m$. Then $N$ is properly embedded in $M$ if $N$ is embedded in $M$, $\partial N \subseteq \partial M$ and $\text{Int}(N) \subseteq \text{Int}(M)$.

Definition 1.3 (Two-sided surface) Let $M$ be an orientable 3-manifold with boundary. A properly embedded connected surface $S \subseteq M$ is two-sided if the complement of $S$ in a sufficiently small neighborhood of $S$ is disconnected. If $S$ is not connected, then $S$ is two-sided if every component of $S$ is two-sided.

If $M$ is an orientable 3-manifold with boundary, then a properly embedded surface $S$ is two-sided if and only if every component of $S$ is orientable.

Definition 1.4 (Incompressible surface) Let $M$ be a compact, orientable 3-manifold with boundary. Then a two-sided, properly embedded surface $S$ is incompressible if the following hold:
1. The inclusion homomorphism $\pi_1(S') \to \pi_1(M)$ is injective for every component $S'$ of $S$;

2. each component of $S$ is distinct from a 2-sphere;

3. the surface $S$ is nonempty.

A non-orientable, properly embedded surface $S$ is incompressible if for any regular neighborhood $N$ of $S$, the two-sided, properly embedded surface $\partial N$ is incompressible.

Let $M$ be an orientable 3-manifold with boundary. Let $S$ be a two-sided, properly embedded surface. Due to the Loop Theorem proven by Papakyriakopoulos, if $\pi_1(S') \to \pi_1(M)$ is not injective for every component $S'$ of $S$, then there is an embedded disk $D \subseteq M$ with $\partial D \subseteq S$ where $\partial D$ is an essential closed curve in $S$. Such a disk is called a compressing disk of $S$.

**Definition 1.5 (Boundary-parallel surface)** Let $M$ be an orientable 3-manifold with boundary. A properly embedded surface $S$ is boundary-parallel if there exists a set $P$ such that $S$ is the frontier of $P$ and the pair $(S, P)$ is homeomorphic to the pair $(S \times \{0\}, S \times [0, 1])$.

Here the frontier of a subset $P$ in an orientable 3-manifold with boundary, $M$, is the intersection of the closure of $P$ with the closure of $M \setminus P$.

**Definition 1.6 (Essential surface)** Let $M$ be a compact, orientable 3-manifold with torus boundary. A properly embedded surface $S$ is essential in $M$ if $S$ is incompressible, and has no boundary-parallel components.

Let $M$ be a compact, orientable 3-manifold with torus boundary and $S$ be an essential surface in $M$ with nonempty boundary. Then the components of $\partial S$ must be pairwise disjoint,
nonseparating closed curves on the boundary torus of $M$. Therefore, the components of $\partial S$ must be pairwise isotopic, essential closed curves in $\partial M$.

**Definition 1.7 (Boundary slope)** Let $M$ be a compact, orientable 3-manifold with torus boundary. If $S$ is an essential surface with nonempty boundary, then the unoriented isotopy class of the components of $\partial S$ is the boundary slope of $S$.

Let $M$ be a compact, orientable 3-manifold with torus boundary and $S$ be an essential surface in $M$ with nonempty boundary. Take a basis $\mu, \lambda$ for $H_1(\partial M, \mathbb{Z})$. Let $\gamma$ be a component of $\partial S$ and give $\gamma$ an orientation. Then $[\gamma] = p\mu + q\lambda \in H_1(\partial M, \mathbb{Z})$ for some integers $p, q$. The boundary slope of $S$ can be represented by $p/q \in \mathbb{Q} \cup \{\infty\}$. Notice that the ratio $p/q$ will be the same regardless of which of the two orientations we choose for $\gamma$.

**Definition 1.8 (Set of boundary slopes)** Let $M$ be a compact, orientable 3-manifold with torus boundary and choose a basis for $H_1(\partial M, \mathbb{Z})$. Then we denote the set of all boundary slopes of essential surfaces in $M$ with respect to this basis by

$$bs(M) = \{p/q \in \mathbb{Q} \cup \{\infty\} \mid p/q \text{ is the boundary slope of an essential surface}\}$$

One fundamental result concerning boundary slopes of essential surfaces is due to A. Hatcher (3). He showed that if $M$ is a compact, orientable 3-manifold with torus boundary, then $bs(M)$ is finite. There has been extensive work done to classify this finite set of boundary slopes of essential surfaces for certain classes of 3-manifolds with torus boundary. A. Hatcher and W. Thurston (4) gave an algorithm for finding $bs(M)$ when $M$ is a 2-bridge knot complement. M.
Culler, W. Jaco and H. Rubinstein (5), and also W. Floyd and A. Hatcher (6) gave algorithms to find $bs(M)$ for $M$ a once-punctured torus bundle. A. Hatcher and U. Oertel (7) did the same for Montesinos knot complements. This algorithm was implemented in a program by N. Dunfield (8). Still there are relatively simple knot complements where the set of boundary slopes of essential surfaces is unknown.
CHAPTER 2

IDEAL TRIANGULATIONS AND CELL DECOMPOSITIONS

Throughout our development of ideal triangulations and spunnormal surfaces we will follow the thorough exposition of Stephan Tillmann (16).

2.1 Pseudo-Manifolds

Let $\Delta$ be a (possibly countably infinite) collection of oriented, polyhedral 3-cells. An orientation on a polyhedral 3-cell is an ordering of its vertices. We also consider the set of subfaces of each $\Delta_i \in \Delta$ as being contained in $\Delta$. All subfaces are taken to be closed. Now let $\Phi$ be a collection of orientation reversing linear homeomorphisms between 2-dimensional faces of $\Delta$ such that each face is the domain or range of at most one map $\phi \in \Phi$.

Definition 2.1 (Pseudo-manifold) Using the notation in the above paragraph, $\mathcal{T}$ is a pseudo-manifold if it is the space obtained by the quotient of $\Delta$ by the maps in $\Phi$ denoted $\mathcal{T} = \Delta/\Phi$. The quotient of a polyhedral 3-cell $\Delta_i \in \Delta$ will be denoted by $\Delta_i \subseteq \mathcal{T}$. The quotient of any subface $\sigma \in \Delta$ will be denoted by $\sigma \subseteq \mathcal{T}$.

Notice that the only possible non-manifold points of such a quotient space are the vertices. The $d$-skeleton of a psuedo-manifold $\mathcal{T}$, denoted $\mathcal{T}^{(d)} \subseteq \mathcal{T}$, is the image of the $d$-dimensional subfaces in $\Delta$ for $0 \leq d \leq 3$. 
2.2 Ideal Triangulations and Cell Decompositions

**Definition 2.2 (Ideal triangulation)** Let $M$ be a compact, oriented 3-manifold with torus boundary. An ideal triangulation of $M$ is a pseudo-manifold, $\mathcal{T} = \tilde{\Delta}/\Phi$, together with a homeomorphism

$$p : \mathcal{T} \setminus \mathcal{T}^{(0)} \to \text{Int}(M).$$

We require that $\tilde{\Delta} = \{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n\}$ is a finite set, and that each 3-cell $\tilde{\Delta}_i \in \tilde{\Delta}$ is a tetrahedron for $1 \leq i \leq n$. We will say the pair $(M, \mathcal{T})$ is an ideal triangulation of a compact, oriented 3-manifold with torus boundary.

Notice that tetrahedra of $\tilde{\Delta}$ are not necessarily embedded in $\text{Int}(M)$ as they may intersect themselves. This allows the tetrahedra and subsets of the tetrahedra to take on very interesting shapes after identification. For interesting examples see the introduction to W. Jaco and J. Rubinstein’s paper (17).

Since $M$ is taken to have only one boundary component, $\mathcal{T}$ has only one vertex, and $\mathcal{T}$ is homeomorphic to $M$ with $\partial M$ collapsed to a point. If $\{\Delta_1, \ldots, \Delta_n\}$ are the tetrahedra in $\mathcal{T}$, then $\mathcal{T}$ must have $n$ edges since $\chi(\text{Int}(M)) = 0$. Due to work by W. Jaco and J. Rubinstein (17) every compact, orientable 3-manifold with torus boundary admits an ideal triangulation. The program SnapPy (18) developed by M. Culler, J. Weeks and N. Dunfield computes ideal triangulations for compact, orientable 3-manifolds with torus boundary.

For convenience, we will give each tetrahedron an orientation (or labeling of its vertices) using the convention of the program SnapPy (18). See Figure 1 below.
Definition 2.3 (Cell Decomposition) Let $M$ be an oriented 3-manifold with boundary. A cell decomposition of $M$ is a pseudo-manifold $T$ together with a homeomorphism

$$p : T \to M.$$ 

We say that the pair $(M, T)$ is a cell decomposition of an oriented 3-manifold with boundary.

Let $(M, T)$ be a cell decomposition of an oriented 3-manifold with boundary. Then every vertex of $T$ must be a manifold point. Also, if $M$ is noncompact, then $T$ must consist of infinitely many 3-cells.

2.3 Polyhedral Complexes

Pseudo-manifolds are examples of more general spaces called polyhedral complexes. Polyhedral complexes play a key role throughout this thesis. Our development of polyhedral complexes,
which will be done as needed throughout the thesis, will follow the exceptional presentation in D. Maclagan and B. Sturmfels’ book (19). Here polyhedra can be of any dimension.

**Definition 2.4 (Polyhedral complex)** Let $\Sigma$ be a collection of convex polyhedra, which are not necessarily compact. We consider faces of elements in $\Sigma$ as being in $\Sigma$. Let $\Phi$ be a collection of linear homeomorphisms between faces of $\Sigma$. Then the quotient space $\Sigma = \Sigma/\Phi$ is a polyhedral complex. The image of any face $\bar{\sigma} \in \Sigma$ will be denoted by $\sigma \subseteq \Sigma$.

We consider polyhedral complexes in the category of piecewise linear spaces, and say that two polyhedral complexes are *isomorphic* if there is a piecewise linear homeomorphism between them.
CHAPTER 3

SPUNNORMAL SURFACES

Throughout this chapter we will continue to follow the development in S. Tillmann's paper (16).

3.1 Normal Disks

Definition 3.1 (Normal arcs and normal disks)

1. A normal arc is a properly embedded arc in the face of a polyhedral 3-cell which has end points in the interiors of distinct edges.

2. A normal disk is a properly embedded disk in a polyhedral 3-cell whose boundary consists of normal arcs and intersects each edge of the polyhedral 3-cell at most once.

Definition 3.2 (Normal isotopy of a polyhedral 3-cell) A normal isotopy of a polyhedral 3-cell Σ is an isotopy of Σ which leaves all of the subcells of Σ fixed.

Most of the development of normal disks in this thesis involves normal disks in the tetrahedra of an ideal triangulation. Therefore, we introduce a few more definitions for normal disks in tetrahedra.
Definition 3.3 (Normal triangles, normal quads and quad-types)

1. A normal disk in a tetrahedron which meets three faces of the tetrahedron is a normal triangle.

2. A normal disk in a tetrahedron which meets four faces of the tetrahedron is a normal quad.

3. In any tetrahedron there are three normal isotopy classes of normal quads called the quad-types. We will denote these quad-types by $Q_1$, $Q_2$, and $Q_3$. See Figure 2.

Notice there are four normal isotopy classes of normal triangles.

Figure 2. The quad-types from left to right: $Q_1$, $Q_2$, $Q_3$. 
3.2 Normal Surfaces and Spunnormal Surfaces

Definition 3.4 (Normal surface with respect to a cell decomposition) Let \((M, \mathcal{T})\) be a cell decomposition of an oriented 3-manifold with boundary, and \(S\) be a properly embedded surface in \(\mathcal{T}\). Then \(S\) is a normal surface with respect to \(\mathcal{T}\) if \(S\) intersects each polyhedral 3-cell of \(\mathcal{T}\) in a collection of normal disks.

Remark 3.5 The normalization process consisting of eight moves introduced in the third chapter of Matveev's book (20) can be used to show that if \(M\) is a compact, orientable 3-manifold with boundary, then any essential surface in \(M\) is isotopic to a normal surface with respect to any cell decomposition \((M, \mathcal{T})\). A property of this normalization process, which will be used later, is that the process never increases the number of intersections of a surface with an edge of \(\mathcal{T}\).

Definition 3.6 (Spunnormal surface with respect to an ideal triangulation) Let \((M, \mathcal{T})\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let \(S\) be a properly embedded surface in \(\mathcal{T} \setminus \mathcal{T}^{(0)}\). Then \(S\) is a spunnormal surface with respect to \(\mathcal{T}\) if \(S\) intersects every tetrahedron of \(\mathcal{T}\) in infinitely many normal disks. See Figure 3 for the intersection of a spunnormal surface with a single tetrahedron.

Remark 3.7 Since normal quads of any two distinct quad-types in a single tetrahedron must intersect, the normal quads of a spunnormal surface must be of the same quad-type in any given tetrahedron.
Figure 3. A possible intersection between a spunnormal surface and a single tetrahedron. There are infinitely many normal triangles, but only finitely many normal quads.

The following is Lemma 1.16 from Tillmann’s paper (16), but we give a proof here to help conceptualize ideal triangulations and spunnormal surfaces.

**Lemma 3.8** Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let $S$ be a spunnormal surface with respect to $\mathcal{T}$. Then $S$ intersects each tetrahedron in $\mathcal{T}$ in finitely many normal quads.

**Proof:** Assume $S$ intersects the tetrahedron $\Delta \subseteq \mathcal{T}$ in infinitely many normal quads. Let $\tilde{\mathcal{T}}$ be the universal cover of $\mathcal{T}$ with the lifted cell decomposition. Then the lift of $S$, $\tilde{S}$, is a properly embedded surface in $\tilde{\mathcal{T}} \setminus \tilde{\mathcal{T}}^{(0)}$. Let $\tilde{\Delta}$ be the lift of $\Delta$ in $\mathcal{T}$.

By Remark 3.7, all of the infinitely many normal quads of $\tilde{S}$ must have the same quad-type $Q$ in $\tilde{\Delta}$. Let $\alpha$ be the line segment connecting the midpoints of the edges of $\tilde{\Delta}$ which normal
quads of quad-type $Q$ do not intersect. Then $\alpha$ must intersect every normal quad of $\tilde{S}$ in $\Delta$. Therefore the normal quads must have an accumulation point on $\alpha$. This contradicts that $\tilde{S}$ is a properly embedded surface in $\tilde{T} \setminus \tilde{T}^{(0)}$. \hfill $\square$

**Definition 3.9 (Vertex linking surface)** Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. A vertex linking surface is an embedded torus in $\mathcal{T} \setminus \mathcal{T}^{(0)}$ which consists of one normal triangle from each normal isotopy class of normal triangles in each tetrahedron of $\mathcal{T}$.

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let $S$ be a vertex linking surface in $\mathcal{T} \setminus \mathcal{T}^{(0)}$. Then $p(S) \subseteq M$ is a boundary-parallel torus where $p : \mathcal{T} \setminus \mathcal{T}^{(0)} \to \text{Int}(M)$ is the homeomorphism from Definition 2.2.

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Tillmann’s Lemma 1.15 (16) shows that any embedded surface in $\mathcal{T}$ that intersects the tetrahedra of $\mathcal{T}$ in only normal triangles is a collection of vertex linking components. By this and Lemma 3.8, a spunnormal surface either has a noncompact component which intersects every neighborhood of the vertex of $\mathcal{T}$; or is a closed, compact surface together with infinitely many vertex linking components. A closed surface in $\mathcal{T}$ with a finite number of components is not considered a spunnormal surface by Definition 3.6.

### 3.3 Associated Properly Embedded Surfaces

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary, and $S$ be a spunnormal surface with respect to $\mathcal{T}$. By 2.3 of N. Dunfield and S. Garoufalidis's
paper (11), we can associate to \( S \) a properly embedded surface in \( M \), \( \tilde{S} \), with every component distinct from a boundary parallel torus. A result of this association is that if \( \tilde{S} \) has nonempty boundary, then it has a well-defined boundary slope whether \( \tilde{S} \) is essential or not. We call the boundary slope of \( \tilde{S} \) the boundary slope of \( S \). We also say that \( S \) is essential if \( \tilde{S} \) is essential in \( M \).

G. Walsh (21) has proven that if \( M \) is a compact, oriented, hyperbolic 3-manifold with boundary the union of tori, then every essential surface in \( M \) which is not a virtual fiber is associated to a spunnormal surface with respect to any ideal triangulation \( \mathcal{T} \) with essential edges. This statement was first conjectured by W. Thurston (13). An embedded surface \( S \) in a oriented, 3-manifold with boundary, \( M \), is a virtual fiber if \( S \) is either a fiber of a fibration of \( M \) over the circle, or a generic fiber of an orbifold-fibration of \( M \) over an interval with mirrored endpoints.

**Definition 3.10 (End of an ideal triangulation)** Let \((M, \mathcal{T})\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let \( v \) be the vertex of \( \mathcal{T} \). Let \( N^\bullet \) be any neighborhood of \( v \) with \( \partial N^\bullet \) a vertex linking surface. Then \( N = N^\bullet \setminus \{v\} \) is an end of \( \mathcal{T} \). Every time we take an end of an ideal triangulation \( N \), \( N^\bullet \) will represent \( N \cup \{v\} \).

Let \((M, \mathcal{T})\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Then any end, \( N \), has a product structure of the form \( \mathbb{T} \times \mathbb{R} \) where \( \mathbb{T} \) is the torus.
Let \((M, T)\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. If \(N\) is an end of \(T\), then there is a homeomorphism \(p_N : (T \setminus N^*) \to M\) which defines a cell decomposition \((M, T \setminus N^*)\) by truncated tetrahedra.

**Remark 3.11** Let \((M, T)\) be an ideal triangulation of a compact, orientable 3-manifold with torus boundary. Let \(S\) be a spunnormal surface with respect to \(T\) and \(N\) be an end of \(T\). For any homeomorphism, \(p_N : (T \setminus N^*) \to M\), \(S\) can be isotoped so that \(p_N(S \cap (T \setminus N^*)) = \tilde{S}\). This follows directly from the construction of \(\tilde{S}\) in 2.3 of N. Dunfield and S. Garoufalidis’s paper (11).

### 3.4 Q-Coordinates

The next proposition is Lemma 1.19 in S. Tillmann’s paper (16). We still provide a proof to help understand how spunnormal surfaces are glued together between tetrahedra.

**Proposition 3.12** Let \((M, T)\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Every spunnormal surface with respect to \(T\) is specified by its normal quads in each tetrahedron of \(T\).

The proof of this proposition will use a few definitions involving normal arcs on a face of a tetrahedron.

Let \(F\) be a triangular face of a tetrahedron. There are three normal isotopy classes of normal arcs in \(F\). Let \(\{v_1, v_2, v_3\} = F^{(0)}\) be the vertices of \(F\). The vertex \(v_i\) for \(1 \leq i \leq 3\) is *dual* to the normal arc \(\alpha\) if one component of \(F \setminus \alpha\) contains \(v_i\) and the other component contains the
other two vertices. Every representative of a normal isotopy class of normal arcs is dual to the same vertex. See Figure 4.

Let \((M, \mathcal{T})\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let \(S\) be a spunnormal surface with respect to \(\mathcal{T}\). Consider the normal disks of \(S\) in the tetrahedra of \(\Delta = \{\Delta_1, \ldots, \Delta_n\}\) before taking the quotient. Focus on one face, \(F\), of a tetrahedron \(\Delta_i\) for \(1 \leq i \leq n\). The normal disks of \(S\) meet \(F\) in a countably infinite collection of normal arcs. A normal arc \(\alpha\) of \(S\) dual to a vertex \(v\) is innermost if the component of \(F \setminus \alpha\) not containing \(v\) does not contain any normal arcs of \(S\) from the same normal isotopy class as \(\alpha\). Up to normal isotopy, there is a well-ordering of the normal arcs of \(S\) in each isotopy class on \(F\). This well-ordering starts with the innermost normal arc for that normal isotopy class, and the successor \(\alpha_n\) of a normal arc \(\alpha_{n-1}\) only has \(\alpha_{n-1}\) and the preceding normal arcs in the component of \(F \setminus \alpha_n\) not containing the dual vertex. See Figure 4.

Let \(F'\) be the face identified to \(F\) by a map in \(\Phi\) where \(\mathcal{T} = \Delta/\Phi\). The normal arcs of one normal isotopy class on \(F\) must be glued to the normal arcs of a corresponding normal isotopy class on \(F'\) agreeing with the well-ordering from above.

*Proof of Proposition 3.12*: Let \(S\) and \(S'\) be two spunnormal surfaces with respect to \(\mathcal{T}\) such that \(S\) and \(S'\) have the same number of normal quads of the same quad-types in each tetrahedron of \(\mathcal{T}\). For any face \(F\) of \(\Delta\) and any normal isotopy class of normal arcs on \(F\), the normal arcs which are on the boundary of normal quads of \(S\) must come first in the well-ordering of normal arcs from this normal isotopy class discussed above. See Figure 3 and Figure 4. Then by the above discussion, the gluing of \(S\) and \(S'\) must agree on the innermost arcs in
each normal isotopy class and continue through the well-ordering of each normal isotopy class of normal arcs. Therefore, the gluing of $S$ and $S'$ is the same on each face in $\mathcal{T}$. Therefore, $S$ and $S'$ are isotopic. □

**Definition 3.13 (Q-coordinate vector)** Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let $\{\Delta_1, \ldots, \Delta_n\}$ be the tetrahedra of $\mathcal{T}$. For any spunnormal surface $S$ the Q-coordinate vector of $S$ is a $3n$-tuple given by

$$(q_{1,1}, q_{2,1}, q_{3,1}, q_{1,2}, \ldots, q_{1,n}, q_{2,n}, q_{3,n}) \in \mathbb{Z}_{\geq 0}^{3n}$$

where each $q_{i,j}$ corresponds to the number of normal quads of $S$ of type $Q_i$ for $1 \leq i \leq 3$ in the tetrahedron $\Delta_j$ for $1 \leq j \leq n$. 

Figure 4. A well-ordering of the normal arcs dual to $v$. The well-ordering continues *ad infinitum.*
By Proposition 3.12 a spunnormal surface is completely determined by its Q-coordinate vector. Also by Remark 3.7, if \((q_{1,1}, q_{2,1}, q_{3,1}, q_{1,2}, ..., q_{3,n})\) is the Q-coordinate vector for a spunnormal surface at most one of \(q_{1,j}, q_{2,j}, q_{3,j}\) can be nonzero for \(1 \leq j \leq n\). All \(3n\)-tuples in \(\mathbb{R}^{3n}_{\geq 0}\) satisfying this last condition are called **admissible**.

### 3.5 Q-Matching Equations

Before we get started, we will need some definitions for general polyhedral complexes.

**Definition 3.14 (Relative interior of a face)** Let \(\Sigma\) be a polyhedral complex and \(\sigma\) be an \(n\)-dimensional face of \(\Sigma\). The relative interior of \(\sigma\), \(r\text{Int}(\sigma)\), is the interior of \(\sigma\) as an \(n\)-dimensional space, not with the subspace topology.

**Definition 3.15 (Star neighborhood, and closed star neighborhood)**

1. Let \(\Sigma\) be a polyhedral complex and \(P\) be a subset of \(\Sigma\). The star neighborhood of \(P\), \(\text{Str}(P)\), is the union of the relative interiors of every face \(\tau\) in \(\Sigma\) such that \(P \cap \tau \neq \emptyset\).

2. The closed star neighborhood of \(P\), \(\overline{\text{Str}}(P)\), is the closure of \(\text{Str}(P)\).

**Definition 3.16 (Abstract neighborhood, equator and hemispheres)** Let \((M, \mathcal{T})\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Consider \(\widetilde{\mathcal{T}}\) the universal cover of \(\mathcal{T}\) with the lifted cell decomposition. An abstract neighborhood of an edge \(e \subseteq \mathcal{T}\), \(B(e) \subseteq \widetilde{\mathcal{T}}\), is the closed star neighborhood of the relative interior of a lifting \(\tilde{e} \subseteq \widetilde{\mathcal{T}}\) of \(e\),

\[ B(e) = \overline{\text{Str}(r\text{Int}(\tilde{e}))}. \]
The abstract neighborhood $B(e)$ is given a cell decomposition by restricting the cell decomposition of $\tilde{T}$. The edges in each tetrahedron of $B(e)$ opposite to $\tilde{e}$ form a closed curve in $\partial B(e)$ which we call the equator of $B(e)$. The complement of the equator in $\partial B(e)$ is separated into two components called hemispheres. See Figure 5.

Figure 5. A picture of the abstract neighborhood, $B(e)$.

Remark 3.17 Let $(M, T)$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Take an edge $e$ in $T$ and consider the abstract neighborhood $B(e)$ in the universal cover $\tilde{T}$ of $T$. Notice that normal triangles in any tetrahedron of $B(e)$ which intersect $\tilde{e}$ do not cross the equator. The only normal disks which intersect $\tilde{e}$ and cross the equator are normal quads.
Definition 3.18 (Edge pairings) Consider a tetrahedron with its vertices labeled as in Figure 2. The edge pairings are the three sets of opposite edges, \( P_1 = \{(0, 1), (2, 3)\} \), \( P_2 = \{(0, 2), (1, 3)\} \), and \( P_3 = \{(1, 2), (0, 3)\} \) where \((a, b)\) is the edge connecting the vertex labeled \(a\) to the vertex labeled \(b\).

Definition 3.19 (Slope of a quad-type) Consider a tetrahedron with its vertices labeled as in Figure 2. For each edge pairing \( P_i \) for \( i \in \{1, 2, 3\} \) we assign a slope to each quad-type by defining the slope function \( s_i : \{Q_1, Q_2, Q_3\} \rightarrow \{-, 0, +\} \) as follows:

\[
s_i(Q_j) = \begin{cases} 
+ & \text{if } j = i \\
- & \text{if } j \equiv i + 1 \mod 3 \\
0 & \text{if } j \equiv i + 2 \mod 3
\end{cases}
\]

See Figure 6.

The output of the slope function \( s_i(Q_j) \) represents the “slope” of a normal quad of quad-type \( Q_j \) at an edge from the edge pairing \( P_i \) when using the right-hand rule. Figure 2 can be used to check that the slope function is consistent for all edge pairings and all quad-types (you may need to tilt your head to check some of them).

Remark 3.20 Let \((M, \mathcal{T})\) be an ideal triangulation of a compact, orientable 3-manifold with torus boundary. Consider an edge \( e \subseteq \mathcal{T} \) and an abstract neighborhood \( B(e) \) in the universal cover \( \tilde{\mathcal{T}} \). Using the slope function for the edge pairing in which \( \tilde{e} \) is contained for each tetrahedron, we see that normal quads from quad-types with + slope cross the equator from one
Figure 6. When $e$ is in the edge pairing $P_i$ and $q$ is of quad-type $Q_j$, $s_i(Q_j) = +$. This represents the slope of $q$ when using the right-hand rule at $e$.

Let $D$ be a disk properly embedded in $B(e)$ which meets every tetrahedron of $B(e)$ in a normal disk and intersects $\bar{e}$. Then $\partial D$ is a closed curve in $\partial B(e)$. Let $x \in \partial D$. Then a path starting at $x$ and following $\partial D$ must eventually come back to $x$. Therefore, by Remark 3.17
and Remark 3.20, if the path intersects \( k \) normal quads from quad-types with + slope, then it must intersect \( k \) normal quads from quad-types with – slope, using the slope function for the edge pairing containing \( \tilde{e} \) in each tetrahedron. Since the lift of any spunnormal surface with respect to \((M, T)\) intersects the abstract neighborhood of some edge in such disks, we get a requirement on normal quads for them to glue up to form a spunnormal surface with respect to \( T \) which is reflected in the following equations.

**Definition 3.21 (Q-matching equations)** Let \((M, T)\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where \(\{\Delta_1, ..., \Delta_n\}\) are the tetrahedra in \(T\). Let \(\{e_1, ..., e_n\}\) be the edges of \(T\). For each edge \(e_k\) with \(1 \leq k \leq n\), we get an equation:

\[
Q_k = \sum_{j=1}^{n} ((a_{k,j} - c_{k,j})q_{1,j} + (b_{k,j} - a_{k,j})q_{2,j} + (c_{k,j} - b_{k,j})q_{3,j}) = 0
\]

where \(a_{k,j}\) is the number of edges in the tetrahedron \(\tilde{\Delta}_j\) from the edge pairing \(P_1\) identified to \(e_k\), \(b_{k,j}\) is the number of edges in the tetrahedron \(\tilde{\Delta}_j\) from the edge pairing \(P_2\) identified to \(e_k\), and \(c_{k,j}\) is the number of edges in the tetrahedron \(\tilde{\Delta}_j\) from the edge pairing \(P_3\) identified to \(e_k\).

These \(n\) equations are called the Q-matching equations.

Let \((M, T)\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Every spunnormal surface with respect to \(T\) has a Q-coordinate vector which is an integral admissible solution to the Q-matching equations.
Definition 3.22 (The admissible solutions to the $Q$-matching equations, $Q(T)$) Let 
$(M, T)$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. The 
admissible solutions to the $Q$-matching equations form a polyhedral complex $Q(T) \subseteq \mathbb{R}^{3n}_{\geq 0}$.

Remark 3.23 Let $(M, T)$ be an ideal triangulation of a compact, oriented 3-manifold with torus 
boundary. For any vector $q \in Q(T)$, the entire ray $\mathbb{R}_{\geq 0} \cdot q$ is contained in $Q(T)$. It is sometimes 
convenient to consider the polyhedral complex $\mathcal{P}Q(T) = Q(T) \cap \{(x_1, \ldots, x_{3n}) \mid x_1 + \cdots + x_{3n} = 1\}$. 
This is a polyhedral complex consisting of rational, compact polytopes. A polytope in $\mathbb{R}^m$ is 
reasonable if the coordinates of every vertex are rational.

Definition 3.24 (Vertex surface) Let $(M, T)$ be an ideal triangulation of a compact, oriented 
3-manifold with torus boundary. Let $q \in \mathbb{R}^{3n}_{\geq 0}$ be a vertex of $\mathcal{P}Q(T)$. Then the spunnormal 
surface with the primitive representative of $\mathbb{R}_{\geq 0} \cdot q$ as its $Q$-coordinate vector is a vertex surface.

Remark 3.25 Let $(M, T)$ be an ideal triangulation of a compact, oriented 3-manifold with torus 
boundary. Consider the $Q$-matching equations as linear polynomials in $\mathbb{R}[q_{1,1}, q_{2,1}, q_{3,1}, q_{2,1}, \ldots, q_{3,n}]$. 
Now choose $n$ coordinates of the form $\{q_{i_1,1}, q_{i_2,2}, \ldots, q_{i_n,n}\}$ where each $i_j \in \{1, 2, 3\}$ for all 
$1 \leq j \leq n$. Then $Q(T)$ restricted to these $n$ coordinates forms a convex polyhedral complex 
since any solution in the restricted coordinates is automatically admissible, and so, the solution 
set is only defined by linear equations.
3.6 Geometric Sum

Definition 3.26 (Compatible spunnormal surfaces) Let \((M, T)\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Two spunnormal surfaces are compatible if they do not have normal quads of different quad-types in any tetrahedron of \(T\).

Definition 3.27 (Geometric sum) Let \((M, T)\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let \(S\) and \(S'\) be two compatible spunnormal surfaces with Q-coordinate vectors \(q\) and \(q' \in Q(T)\). The geometric sum of \(S\) and \(S'\) is the surface \(S \uplus S'\) whose Q-coordinate vector is \(q + q' \in Q(T)\).

This geometric sum operation is well-defined. The geometric sum operation adds a great deal of structure to the theory of spunnormal surfaces. However, we will just need what has been stated for the main result in this thesis. For a thorough presentation see S. Tillmann’s paper (16).
CHAPTER 4

THE DEFORMATION VARIETY

4.1  Ideal Tetrahedra

Definition 4.1 (Ideal tetrahedron in \( \mathbb{H}^3 \)) Let \( \mathbb{H}^3 = \mathbb{H}^3 \cup S^2_{\infty} \) where \( S^2_{\infty} \) is the sphere at infinity. An ideal tetrahedron in \( \mathbb{H}^3 \) is a tetrahedron contained in \( \mathbb{H}^3 \) with vertices on \( S^2_{\infty} \).

Let \( \Delta \subseteq \mathbb{H}^3 \) be an ideal tetrahedron with the SnapPy orientation (see Figure 1) in the upper half-space model of \( \mathbb{H}^3 \) with \( S^2_{\infty} \) identified to \( \mathbb{C} = \mathbb{C} \cup \{ \infty \} \). Apply an orientation preserving isometry to \( \mathbb{H}^3 \) such that the vertex labeled 0 in the SnapPy orientation is sent to 0 \( \in \mathbb{C} \), the vertex labeled 1 in the SnapPy orientation is sent to \( \infty \in \mathbb{C} \), and the vertex labeled 3 in the SnapPy orientation is sent to 1 \( \in \mathbb{C} \). The vertex labeled 2 in the SnapPy orientation must then be mapped to some complex number \( z \in \mathbb{C} \setminus \{0, 1\} \).

Definition 4.2 (Shape parameters) Let \( \Delta \) be an ideal tetrahedron in \( \mathbb{H}^3 \) with the above identification of vertices. Shape parameters are assigned to the edges of \( \Delta \) as follows. Assign the shape parameter \( z \) to the edge \((0, \infty)\) of \( \Delta \), assign the shape parameter \( 1/(1-z) \) to the edge \((1, \infty)\), and assign the shape parameter \( (z-1)/z \) to the edge \((z, \infty)\). Opposite edges in \( \Delta \) are assigned the same shape parameter. In terms of the edge pairings of Definition 3.18, the edges of \( P_1 \) are assigned the shape parameter \( z \), the edges of \( P_2 \) are assigned the shape parameter \( 1/(1-z) \), and the edges of \( P_3 \) are assigned the shape parameter \( (z-1)/z \). See Figure 7.
These shape parameters are the cross-ratios for the dihedral angles of each edge. Opposite edges of $\Delta$ are assigned the same shape parameter since the dihedral angles of opposite edges of an ideal tetrahedron in $\mathbb{H}^3$ are equal. Figure 7 shows an ideal tetrahedron in $\mathbb{H}^3$ with the vertices labeled and shape parameters given to one edge from each edge pair.

![Figure 7. An ideal tetrahedron in $\mathbb{H}^3$ with labeling.](image)

### 4.2 Gluing Equations and the Deformation Variety

Let $(M, T)$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra in $T$ with the SnapPy orientation, see Figure 1. Each
tetrahedron $\Delta_i$ for $1 \leq i \leq n$ can be identified with an ideal tetrahedron in $\mathbb{H}^3$ by an orientation preserving isometry as above, and so be assigned shape parameters for its edges:

$$z_i, \frac{1}{1 - z_i}, \frac{z_i - 1}{z_i}.$$  

See Figure 7.

Let $\{e_1, ..., e_n\}$ be the edges of $T$. Take an edge $e_k$ for $1 \leq k \leq n$. Consider the abstract neighborhood of $e$, $B(e)$, with the lifted hyperbolic structure. The tetrahedra of $B(e)$ must glue up around $\tilde{e}_k$. Therefore, the product of the shape parameters assigned to $\tilde{e}_k$ from each tetrahedron in $B(e)$ must be 1. These equations are made explicit in the next definition.

**Definition 4.3 (Gluing equations)** Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra in $\mathcal{T}$. Let $\{e_1, ..., e_n\}$ be the edges of $\mathcal{T}$. For each edge $e_k$ with $1 \leq k \leq n$ we get an equation:

$$R_k = \prod_{i=1}^{n} z_i^{a_{k,j}} \left( \frac{1}{1 - z_i} \right)^{b_{k,j}} \left( \frac{z_i - 1}{z_i} \right)^{c_{k,j}} = 1$$

where $a_{k,j}$ is the number of edges in the tetrahedron $\tilde{\Delta}_j$ from the edge pairing $P_1$ identified to $e_k$, $b_{k,j}$ is the number of edges in the tetrahedron $\tilde{\Delta}_j$ from the edge pairing $P_2$ identified to $e_k$, and $c_{k,j}$ is the number of edges in the tetrahedron $\tilde{\Delta}_j$ from the edge pairing $P_3$ identified to $e_k$. Because the product $R_1 \cdots R_n = 1$ for all $(z_1, ..., z_n) \in (\mathbb{C} \setminus \{0, 1\})^n$, we omit the $n$th equation and consider $\{R_1, ..., R_{n-1}\}$ the gluing equations for $(M, \mathcal{T})$. 

Notice that the gluing equations could be defined with any edge pair receiving the shape parameter $z_i$. However, the result proven in this thesis is better presented when this convention is set.

**Definition 4.4 (Augmented Gluing Equations)** Let $(M, T)$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, \ldots, \Delta_n\}$ are the tetrahedra in $T$. Let $\{e_1, \ldots, e_n\}$ be the edges of $T$. The augmented gluing equations are the gluing equations with the substitution $w_i = 1 - z_i$. Again we get one equation for each edge $e_k$ where $1 \leq k \leq n - 1$:

$$R_k = \prod_{i=1}^{n} (-1)^{c_{k,j}} z_i^{a_{k,j} - c_{k,j}} w_i^{c_{k,j} - b_{k,j}} = 1$$

where $a_{k,j}$ is the number of edges in the tetrahedron $\Delta_j$ from the edge pairing $P_1$ identified to $e_k$, $b_{k,j}$ is the number of edges in the tetrahedron $\Delta_j$ from the edge pairing $P_2$ identified to $e_k$, and $c_{k,j}$ is the number of edges in the tetrahedron $\Delta_j$ from the edge pairing $P_3$ identified to $e_k$.

The following example gives the augmented gluing equations for the 5$_2$-knot complement. This will be a running example throughout the remainder of the thesis.
Example 4.5 (5_2-knot complement) The ideal triangulation from SnapPy for the 5_2-knot complement uses three tetrahedra. The augmented gluing equations for the 5_2-knot complement are the following:

\[ R_1 = w_1 \cdot z_2^{-1} \cdot z_3 \cdot w_3^{-2} = 1, \]
\[ R_2 = (-1) \cdot z_1^{-1} \cdot z_2 \cdot w_2^{-1} \cdot z_3^{-2} \cdot w_3^{-2} = 1. \]

(4.1)

Let \((M, T)\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where \(\{\Delta_1, ..., \Delta_n\}\) are the tetrahedra of \(T\). Notice that the augmented gluing equations can only have a zero in the denominator if \(z_i\) or \(w_i\) is zero for some \(i \in \{1, ..., n\}\). However, any ideal tetrahedron cannot have a shape parameter with \(z_i = 1\) or \(z_i = 0\). Therefore, the augmented gluing equations can be expressed as polynomial equations. For \(1 \leq k \leq n - 1\), let \(R'_k = 0\) be the polynomial equations coming from the equation \(R_k = 1\).

Example 4.6 (5_2-knot complement) The polynomial augmented gluing equations for the 5_2-knot complement are:

\[ R'_1 = w_1z_3 - z_2w_3^2 = 0, \]
\[ R'_2 = z_2w_3^2 + z_1w_2z_3^2 = 0. \]
Definition 4.7 (Deformation Variety) Let \((M, \mathcal{T})\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where \(\{\Delta_1, ..., \Delta_n\}\) are the tetrahedra in \(\mathcal{T}\). Let

\[
\mathcal{AG} = \{R'_1, ..., R'_{n-1}, z_1 + w_1 - 1, z_2 + w_2 - 1, ..., z_n + w_n - 1\} \subseteq \mathbb{C}[z_1, ..., z_n, w_1, ..., w_n]
\]

be the set of polynomial augmented gluing equations together with the linear equations \(w_i = 1 - z_i\) for \(1 \leq i \leq n\). Then the deformation variety, \(\mathcal{D}(\mathcal{T})\) is given by

\[
\mathcal{D}(\mathcal{T}) = V(\langle \mathcal{AG} \rangle) \subseteq (\mathbb{C} \setminus \{0, 1\})^{2n}
\]

where \(V(\langle \mathcal{AG} \rangle)\) is the zero set of the ideal generated by \(\mathcal{AG}\).

Notice that the deformation variety from an ideal triangulation of a compact, oriented 3-manifold with torus boundary is not necessarily irreducible.

Remark 4.8 Let \((M, \mathcal{T})\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. For the main result of this thesis we will need to assume every irreducible component of \(\mathcal{D}(\mathcal{T})\) is 1-dimensional. This is true in examples, and is thought to be true in general for such an \((M, \mathcal{T})\).

4.3 The Character Variety

Definition 4.9 (Character variety) Let \(M\) be a compact, oriented 3-manifold with torus boundary. The character variety of \(M\), \(X(M)\), is the \(SL_2(\mathbb{C})\) character variety of \(\pi_1(M)\).

The following is shown in N. Dunfield and S. Garoufalidis’s paper as Lemma 3.5 (11).
Proposition 4.10 Let $(M, T)$ be an ideal triangulation of a compact, orientable 3-manifold with torus boundary. There is a regular map from $\mathcal{D}(T)$ to $\mathcal{X}(M)$ where $\mathcal{X}(M)$ is the $\text{PSL}_2(\mathbb{C})$ character variety of $\pi_1(M)$.

This regular map is usually called the pseudo-developing map and is not surjective.

Definition 4.11 (Ideal points of a variety) Let $V$ be an affine variety and $\tilde{V}$ be the projective completion of $V$. Then $\tilde{V} \setminus V$ is the set of ideal points of $V$.

Let $(M, T)$ be an ideal triangulation of a compact, orientable 3-manifold with boundary. M. Culler and P. Shalen (14) have shown that ideal points of the character variety $X(M)$ correspond to essential surfaces in $M$. Also, ideal points of $\mathcal{X}(M)$ can be associated to ideal points of $X(M)$. However, not every ideal point of $\mathcal{D}(T)$ maps to an ideal point of $X(M)$ using this association and the map from Proposition 4.11.

4.4 Associated Spunnormal Surfaces

Let $(M, T)$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, \ldots, \Delta_n\}$ are the tetrahedra in $T$. Let $D$ be an irreducible curve of $\mathcal{D}(T)$. For an ideal point $\xi$ of $D$ we can associate a spunnormal surface $S(\xi)$ with Q-coordinate vector $(q_{1,1}, \ldots, q_{3,n})$ as follows. For each tetrahedron $\Delta_i$ for $1 \leq i \leq n$ define the entries $(q_{1,i}, q_{2,i}, q_{3,i})$ as follows. Let $z_i$ be the shape parameter of the edge $(0, \infty)$ in $\Delta_i$. The projective completion of $D$, $\widetilde{D}$, is a Riemannian surface, and $z_i$ is a function of this surface. Let $v_\xi$ be the discrete valuation.
defined by $\xi$ on the function field of $\bar{D}$. Then the coordinates $(q_{1,i}, q_{2,i}, q_{3,i})$ are assigned using the following:

\[
(q_{1,i}, q_{2,i}, q_{3,i}) = \begin{cases} 
(v_\xi(z_i), 0, 0) & \text{if } v_\xi(z_i) > 0 \\
(0, -v_\xi(z_i), 0) & \text{if } v_\xi(z_i) < 0 \\
(0, 0, v_\xi(1 - z_i)) & \text{if } v_\xi(z_i) = 0 \text{ and } z_i(\xi) \notin \mathbb{C} \setminus \{0, 1\} \\
(0, 0, 0) & \text{if } z_i(\xi) \in \mathbb{C} \setminus \{0, 1\}.
\end{cases}
\]

The fact that the shape parameters around an edge in $\mathcal{T}$ must multiply to 1, means that the valuation of the shape parameters at $\xi$ must sum to zero around the edge. This gives exactly that the Q-coordinate vector for $S(\xi)$ satisfies the Q-matching equations, and that $S(\xi)$ glues up on the faces to give a spunnormal surface. We say $S(\xi)$ is the spunnormal surface associated to the ideal point $\xi$.

**Definition 4.12 (Reduction of a properly embedded surface)** Let $M$ be a 3-manifold with boundary. A properly embedded surface $S$ in $M$ is said to reduce to $S'$ if there is a sequence of compressions along compressing disks, boundary compressions, elimination of 2-spheres, and elimination of boundary parallel components which turns $S$ into $S'$. We then say that $S'$ is a reduction of $S$.

This next result can be found in N. Dunfield and S. Garoufalidis’s paper as Theorem 3.8 (11) and S. Tillmann’s paper as Proposition 4.3 (22). This result uses the pseudo-developing map between the deformation variety and the $\text{PSL}_2(\mathbb{C})$ character variety of the fundamental group of the manifold.
Proposition 4.13 Let $(M, T)$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let $\xi$ be an ideal point of an irreducible curve $D \subseteq D(T)$. Suppose $S$ is the spunnormal surface with respect to $T$ which is associated to $\xi$. Also assume the associated properly embedded surface in $M$, $\hat{S}$, is two-sided and has nonempty boundary with boundary slope $\alpha$. Then any reduction of $\hat{S}$ has nonempty boundary with boundary slope $\alpha$. In particular, $\hat{S}$ can be reduced to a nonempty essential surface in $M$ with boundary slope $\alpha$.

Let $(M, T)$ be an ideal triangulation of a compact, orientable 3-manifold with torus boundary. Proposition 4.14 tells us the boundary slope of a spunnormal surface associated to an ideal point of $D(T)$ is a boundary slope in $\text{bs}(M)$. 
CHAPTER 5

TROPICAL GEOMETRY

This chapter follows the presentation of concepts in tropical geometry in the paper by T. Bogart, A. Jensen, D. Speyer, B. Sturmfels and R. Thomas (23). Please see their paper for details.

5.1 Tropical Varieties

Definition 5.1 (w-weight, and initial form) Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial.

1. Let \( w \in \mathbb{R}^n \) be a vector. Then the \( w \)-weight of a term \( cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \) in \( f \), where \( a_i \in \mathbb{Z}_{\geq 0} \) for \( 1 \leq i \leq n \), is defined as \( w \cdot (a_1, \ldots, a_n) \in \mathbb{R} \).

2. Let \( w \in \mathbb{R}^n \) be a vector. The initial form of \( f \) with respect to \( w \) denoted \( \text{in}_w(f) \) is the sum of all terms in \( f \) of lowest \( w \)-weight.

Example 5.2 (5_2-knot complement) Take the first polynomial augmented gluing equation from the 5_2-knot complement,

\[ R'_1 = w_1z_3 - z_2w_3^2 = 0. \]

Consider \( R'_1 \in \mathbb{C}[z_1, w_1, z_2, w_2, z_3, w_3] \). Let \( w_1 = (0, 1, 1, 2, 2, 1) \). Then the \( w_1 \)-weight of the term \( w_1z_3 \) is 3. The \( w_1 \)-weight of the term \( -z_2w_3^2 \) is also 3. Therefore, the initial form of \( R'_1 \) with respect to \( w_1 \) is

\[ \text{in}_{w_1}(R'_1) = w_1z_3 - z_2w_3^2. \]
Now let $w_2 = (-2, -2, -1, -1, 1, 0)$. Then the $w_2$-weight of the term $w_1z_3$ is $-1$. The $w_2$-weight of the term $-z_2w_3^2$ is also $-1$. So

$$\text{in}_{w_2}(R'_1) = w_1z_3 - z_2w_3^2.$$ 

Finally, let $w_3 = (4, 1, -1, 0, -2, 3)$. Then the $w_3$-weight of the term $w_1z_3$ is $-1$. The $w_3$-weight of the term $-z_2w_3^2$ is $5$. So

$$\text{in}_{w_3}(R'_1) = w_1z_3.$$ 

**Definition 5.3 (Tropical hypersurface of a polynomial)** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. The tropical hypersurface of $f$ is the set

$$\text{Trop}(f) = \{ w \in \mathbb{R}^n \mid \text{in}_w(f) \text{ is not a monomial} \}.$$ 

A tropical hypersurface can be thought of as the solutions to a tropical linear equation, where addition is defined by $a \oplus b = \min\{a, b\}$, and multiplication is defined by $a \odot b = a + b$.

**Example 5.4 (5_2-knot complement)** Take the first polynomial augmented gluing equation from the 5_2-knot complement,

$$R'_1 = w_1z_3 - z_2w_3^2 = 0.$$
Consider $R_1' \in \mathbb{C}[z_1, w_1, z_2, w_2, z_3, w_3]$. Let $w_1 = (0, 1, 1, 0, 2, 1)$, $w_2 = (1, 0, 1, 0, -1, -1)$, and $w_3 = (4, 1, -1, 0, -2, 3)$ as in Example 5.2. Then, by Example 5.2, $w_1 \in \text{Trop}(R_1')$, and $w_2 \in \text{Trop}(R_1')$, however $w_3 \notin \text{Trop}(R_1')$.

**Definition 5.5 (Newton polytope of a polynomial)** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ where

$$f = \sum_{i=1}^{m} c_i x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}.$$

Let $V = \{(a_{1,i}, a_{2,i}, \ldots, a_{n,i}) \in \mathbb{R}^n \mid 1 \leq i \leq m\}$ be the set of exponent vectors for each term of $f$. Then the Newton polytope of $f$, $N(f)$, is the polyhedron formed by the convex hull of $V$ in $\mathbb{R}^n$.

**Remark 5.6** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Then the tropical hypersurface of $f$, $\text{Trop}(f)$, is the inner pointing normal vectors of each nonvertex subcell of the Newton polytope $N(f)$. A vector $w \in \text{Trop}(f)$ must be normal to a nonvertex subcell of $N(f)$ since this guarantees that two or more terms of $f$ have the same $w$-weight. A vector $w \in \text{Trop}(f)$ which is normal to a nonvertex subcell of $N(f)$ must also be inner pointing since this guarantees that the terms corresponding to the vertices of the subcell to which $w$ is normal are of lowest $w$-weight.

**Remark 5.7** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. The tropical hypersurface $\text{Trop}(f)$ is a polyhedral complex such that, if $w \in \text{Trop}(f)$ then $\mathbb{R}_+ \cdot w \subseteq \text{Trop}(f)$. Therefore, it is often convenient to consider the projection of $\text{Trop}(f)$ onto the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. 
Example 5.8 (5₂-knot complement) Consider the first polynomial augmented gluing equation for the 5₂-knot complement,

\[ R'_1 = w_1 z_3 - z_2 w_3^2 = 0. \]

In order to get a picture of a Newton polytope and tropical hypersurface we make the substitution

\[ 1 - z_i = w_i. \]

We get the polynomial gluing equation,

\[ (1 - z_1) z_3 - z_2 (1 - z_3)^2 = 0. \]

Then the set \( V \) from Definition 5.5 is given by

\[ V = \{ (0, 0, 1), (1, 0, 1), (0, 1, 0), (0, 1, 1), (0, 1, 2) \}. \]

The Newton polytope \( N(R'_1) \subseteq \mathbb{R}^3 \) is pictured in Figure 8 by the bold lines. Then we can compute the inner pointing normals for each edge and face of \( N(R'_1) \). This gives us \( \text{Trop}(R'_1) \).

The tropical hypersurface \( \text{Trop}(R'_1) \) projected onto the unit sphere is pictured in Figure 9 by the bold arcs. The vertices are labeled by the primitive integral vectors of the rays which they represent.

By Remark 5.7, tropical hypersurfaces fall into a special category of polyhedral complexes called fans which we define now.
Definition 5.9 (Fan) A fan is a polyhedral complex where every face is a cone emanating from a common vertex. We use the terms either face or cone to refer to faces of a fan. The term ray may be used to refer to a 1-dimensional cone.

Remark 5.10 Let \((M, T)\) be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Then, by Remark 3.23, the set of admissible solutions to the Q-matching equations, \(Q(T)\), is a fan emanating from the origin in \(\mathbb{R}^{3n}\) with the origin represented by a union of infinitely many vertex linking tori.
Figure 9. A picture of Trop($\mathcal{R}_1$) projected onto the unit sphere represented by the bold arcs with vertices labeled by primitive integral representatives of the ray.

**Definition 5.11 (Tropical prevariety)** Let $\{f_1, ..., f_m\} \subseteq \mathbb{C}[x_1, ..., x_n]$ be a finite collection of polynomials. Then the tropical prevariety of $\{f_1, ..., f_m\}$, $\text{PreTrop}(\{f_1, ..., f_m\})$, is the intersection of the tropical hypersurfaces of each $f_i$ for $1 \leq i \leq m$. That is,

$$\text{PreTrop}(\{f_1, ..., f_m\}) = \text{Trop}(f_1) \cap \cdots \cap \text{Trop}(f_m).$$
The finite intersection of fans emanating from the same point is again a fan emanating that point. Therefore, if \( \{f_1, \ldots, f_m\} \subseteq \mathbb{C}[x_1, \ldots, x_n] \), then the tropical prevariety, \( \text{PreTrop}(\{f_1, \ldots, f_m\}) \), is a fan.

**Example 5.12 (5_2-knot complement)** Let

\[
\overline{R}_2 = z_2 - 2z_2z_3 + z_2z_4^2 + z_1z_3^2 - z_1z_2z_3^2 = 0
\]

be second polynomial gluing equation for the 5_2-knot complement with the terms expanded. The Newton polytope for \( \overline{R}_2 \) is given in Figure 10 by the bold lines. The tropical hypersurface for \( \overline{R}_2 \) projected onto the unit sphere is then given in Figure 11 by the bold arcs. If we consider the intersection of \( \text{Trop}(\overline{R}_1) \) with \( \text{Trop}(\overline{R}_2) \), we see that the tropical prevariety, \( \text{PreTrop}(\{\overline{R}_1, \overline{R}_2\}) \) projected onto the unit sphere, is given by the bold arc and four points in Figure 12.

**Definition 5.13 (Tropical Variety)** Let \( I \subseteq \mathbb{C}[x_1, \ldots, x_n] \) be an ideal. The tropical variety of \( I \) denoted \( \text{Trop}(I) \) is the intersection of the tropical hypersurfaces \( \text{Trop}(f) \) as \( f \) runs over all polynomials in \( I \).

Notice that for \( f \in \mathbb{C}[x_1, \ldots, x_n] \), the tropical hypersurface \( \text{Trop}(f) \) is equal to the tropical variety \( \text{Trop}(\langle f \rangle) \).

**Example 5.14 (5_2-knot complement)** A. Jensen has written a program that utilizes the algorithm created by T. Bogart, A. Jensen, D. Speyer, B. Sturmfels and R. Thomas (23) to
compute tropical varieties called gfan (24). Using gfan, we find that the vectors $(1, 1, -1)$, $(1, -1, 0)$, $(0, 2, 1)$, $(-1, 0, -1)$ and $(-2, -1, 1)$ are in $\text{Trop}((\overline{R_1}, \overline{R_2}))$. See Figure 13.

5.2 Tropical Bases

**Theorem 5.15** For every ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, the tropical variety of $I$ is the tropical prevariety of some finite set $\{f_1, \ldots, f_m\} \subseteq \mathbb{C}[x_1, \ldots, x_n]$.

T. Bogart, A. Jensen, D. Speyer, B. Sturmfels and R. Thomas give an algorithm for computing such a set for any ideal (23).
Definition 5.16 (Tropical basis) Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$. If $\{f_1, \ldots, f_m\} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a set such that $\text{Trop}(I) = \text{PreTrop}(\{f_1, \ldots, f_m\})$ and $I = \langle f_1, \ldots, f_m \rangle$, then $\{f_1, \ldots, f_m\}$ is a tropical basis for $I$.

Remark 5.17 By Theorem 5.15, every ideal has a tropical basis. Every tropical variety is then the intersection of finitely many fans and is therefore a fan itself.
Figure 12. A picture of $\text{PreTrop}(\{\mathcal{R}'_1, \mathcal{R}'_2\})$ projected onto the unit sphere represented by the bold arcs and dots with vertices labeled by primitive integral representatives of the ray.

Example 5.18 (52-knot complement) The tropical basis found by gfan for $\langle \mathcal{R}'_1, \mathcal{R}'_2 \rangle$ is

$$\{-z_2z_3^2 - z_1z_3 + 2z_2z_3 - z_2 + z_3, \quad -z_1z_2z_3^2 + z_1z_3^2 + z_2z_3^2 - 2z_2z_3 + z_2, \quad z_1z_2z_3^2 - z_1z_3^2 + z_1z_3 - z_3\}$$
5.3 The Fundamental Theorem of Tropical Geometry

Definition 5.19 (Puiseux series) A Puiseux series is a series of the form

\[ p(t) = c_1 t^{a_1} + c_2 t^{a_2} + \ldots \]
where \( c_i \in \mathbb{C} \), \( c_1 \neq 0 \) unless \( p \) is identically 0, \( a_i \in \mathbb{Q} \), \( a_i \leq a_{i+1} \), and the \( a_i \) have a common denominator. All conditions are for all \( i \geq 1 \). The set of Puiseux series creates an algebraically closed field \( \mathbb{C}\{\{t\}\} \). This field comes with a valuation

\[ v : \mathbb{C}\{\{t\}\}^* \rightarrow \mathbb{Q} \]

given by

\[ c_1 t^{a_1} + c_2 t^{a_2} + ... \mapsto a_1. \]

**Theorem 5.20 (Fundamental Theorem of Tropical Geometry)** Let \( I \subseteq \mathbb{C}[x_1, \ldots, x_n] \) be an ideal. Then

\[ \text{Trop}(I) = \{(v(p_1), v(p_2), \ldots, v(p_n)) \in \mathbb{Q}^n \mid (p_1, \ldots, p_n) \in P(I)\} \]

where \( P(I) \) is the set of all \( n \)-tuples of Puiseux series, \( (p_1, \ldots, p_n) \in (\mathbb{C}\{\{t\}\}^*)^n \) such that \( (p_1(t), p_2(t), \ldots, p_n(t)) \in V(I) \) for every \( t \) where the \( p_i(t) \) are defined.

### 5.4 Proper Intersections

**Definition 5.21 (Dimension of a fan)** Let \( \Sigma \) be a fan. The dimension of \( \Sigma \), \( \dim(\Sigma) \), is the maximum of the dimensions of the cones in \( \Sigma \).

**Definition 5.22 (Proper intersection)** Let \( I \) and \( I' \) be two ideals in \( \mathbb{C}[x_1, \ldots, x_n] \). We say \( \text{Trop}(I) \) and \( \text{Trop}(I') \) intersect properly at a point \( w \in \mathbb{R}^n \) if \( \text{Trop}(I) \cap \text{Trop}(I') \) has codimension equal to \( \text{codim}(V(I)) + \text{codim}(V(I')) \) in a neighborhood of \( w \).
The following is shown by B. Osserman and S. Payne as Theorem 1.1 (15).

**Theorem 5.23 (Osserman and Payne)** Let $I$ and $I'$ be two ideals in $\mathbb{C}[x_1, \ldots, x_n]$. Suppose $\text{Trop}(I)$ and $\text{Trop}(I')$ intersect properly at $w$. Then $w$ is in $\text{Trop}(I + I')$.

**Definition 5.24 (Isolated ray)** Let $\Sigma$ be a fan and $R$ be a $1$-dimensional cone of $\Sigma$. Then $R$ is an isolated ray if $R$ is not a subcone of any higher dimensional cones.

The following is a consequence of Theorem 5.23.

**Proposition 5.25** Let $\{P_1, \ldots, P_{m-1}\} \subseteq \mathbb{C}[x_1, \ldots, x_m]$ where every irreducible component of $V(\langle P_1, \ldots, P_{m-1} \rangle) \subseteq \mathbb{C}^m$ is $1$-dimensional. Then if $w \in \mathbb{R}^m$ is contained in an isolated ray of $\text{PreTrop}(\{P_1, \ldots, P_{m-1}\})$, then $w \in \text{Trop}(\langle P_1, \ldots, P_{m-1} \rangle)$.

**Proof:** If $w$ is the origin, then $w$ is clearly in $\text{Trop}(\langle P_1, \ldots, P_{m-1} \rangle)$. So assume $w$ is not the origin. We will show for every $1 \leq j \leq m-1$, the open star neighborhood of $w$, $\text{Str}_j(w) \subseteq \text{PreTrop}(\{P_1, \ldots, P_j\})$ is contained in $\text{Trop}(\langle P_1, \ldots, P_j \rangle)$ by induction on $j$. The base case trivial. Now assume $\text{Str}_{j-1}(w) \subseteq \text{PreTrop}(\{P_1, \ldots, P_{j-1}\})$ is also contained in $\text{Trop}(\langle P_1, \ldots, P_{j-1} \rangle)$. Now,

$$\text{Str}_j(w) \subseteq \text{Str}_{j-1}(w) \subseteq \text{Trop}(\langle P_1, \ldots, P_{j-1} \rangle)$$

and

$$\text{Str}_j(w) \subseteq \text{Trop}(P_j).$$

Therefore, $\text{Str}_j(w)$ is in the intersection of these two tropical varieties. Also, $\text{Str}_j(w)$ has codimension $m - j$ since we must cut down the dimension of $\text{Str}_{i-1}(w)$ by one each time
we intersect with \( \text{Trop}(P_i) \) for \( 1 < i \leq m - 1 \) in order to end with \( \text{Str}_{m-1}(w) \) equal to an isolated ray. Therefore, every point of \( \text{Str}_j(w) \) is a proper intersection of \( \text{Trop}((P_1, ..., P_{j-1})) \) and \( \text{Trop}(P_j) \) since we must cut down the dimension of \( V((P_1, ..., P_{i-1})) \) by one each time we intersect with \( V((P_i)) \) for \( 1 < i \leq m - 1 \) in order to end with every irreducible component of \( V((P_1, ..., P_{m-1}) \) being 1-dimensional. Then \( \text{Str}_j(w) \subseteq \text{Trop}((P_1, ..., P_{j-1}) + (P_j)) \) by the theorem of Osserman and Payne (15), Theorem 5.23 in this thesis. Finally \( \text{Trop}((P_1, ..., P_{j-1}) + (P_j)) = \text{Trop}((P_1, ..., P_j)) \). □

**Example 5.26** (52-knot complement) Just by looking at the tropical prevariety \( \text{PreTrop}([\mathcal{R}_1, \mathcal{R}_2]) \) (see Figure 12), Proposition 5.25 shows that

\[
\{(−2,−1,1),(0,2,1),(1,1,−1), (−1,0,−1)\} \subseteq \text{Trop}((\mathcal{R}_1', \mathcal{R}_2'))
\]

Therefore, by the Fundamental Theorem of Tropical Geometry (Theorem 5.20), these vectors correspond to ideal points of \( \mathcal{D}(\mathcal{T}) \). However, there are some difficulties with using the regular gluing equations and not the augmented gluing equations.

For example, Let \( \xi \) be the ideal point of \( \mathcal{D}(\mathcal{T}) \) corresponding to the the vector (0, 2, 1), and \( v_\xi \) be the associated valuation. We know that \( v_\xi(z_2) = 2 \) and \( v_\xi(z_3) = 1 \). Therefore, the associated surface should have 2 normal quads of quad-type \( Q_1 \) in the tetrahedron \( \Delta_2 \), and 1 normal quad of quad-type \( Q_1 \) in the tetrahedron \( \Delta_3 \). However, the zero in the first coordinate only tells \( v_\xi(z_1) = 0 \). This could represent \( z_1 \to 1 \) as we approach the ideal point, and we should be
assigning some number of normal quads of type $Q_3$ to $\Delta_1$; or $z_1 \to x \in \mathbb{C} \setminus \{0,1\}$ in which case no quads should be assigned to $\Delta_1$.

Even if a spunnormal surface can be assigned to an ideal point, then it still is unknown whether that spunnormal surface is actually essential. Proposition 4.14 does tell us that its boundary slope is the boundary slope of an essential surface. The main theorem of this thesis will give sufficient conditions for a spunnormal surface corresponding to an ideal point to be essential.
CHAPTER 6

MAIN RESULT

In this chapter we prove the main theorem of the thesis. However, before the main theorem, we prove a result which is in S. Tillmann’s paper (22) as Proposition 3.1.

Proposition 6.1 Let \((M, \mathcal{T})\) be an ideal triangulation of a compact, orientable 3-manifold with torus boundary where \(\{\Delta_1, \ldots, \Delta_n\}\) are the tetrahedra of \(\mathcal{T}\). Let

\[
\mathcal{AG} = \{R'_1, \ldots, R'_{n-1}, z_1 + w_1 - 1, \ldots, z_n + w_n - 1\} \subseteq \mathbb{C}[z_1, w_1, \ldots, z_n, w_n]
\]

where \(\{R'_1, \ldots, R'_{n-1}\}\) are from the polynomial augmented gluing equations for \((M, \mathcal{T})\). Then \(\text{PreTrop}(\mathcal{AG})\) and \(Q(\mathcal{T})\) are isomorphic as piecewise linear spaces.

Proof: We define a map \(\varphi : Q(\mathcal{T}) \rightarrow \text{PreTrop}(\mathcal{AG})\) given by

\[
\varphi((q_1,1, q_2,1, q_3,1, \ldots, q_1,n, q_2,n, q_3,n)) = (x_1, y_1, \ldots, x_n, y_n)
\]

where

\[
(x_i, y_i) = \begin{cases} 
(q_1, i, 0) & \text{if } q_1, i > 0 \\
(-q_2, i, -q_2, i) & \text{if } q_2, i > 0 \\
(0, q_3, i) & \text{if } q_3, i > 0 \\
(0, 0) & \text{if } q_1, i = q_2, i = q_3, i = 0
\end{cases}
\]
Then $\varphi$ is well-defined since any $q \in Q(\mathcal{T})$ must be admissible.

Also, for each $(x_i, y_i)$ from a vector in the range of $\varphi$ for $1 \leq i \leq n$, either

1. $x_i = y_i$, and $x_i, y_i \leq 0$,
2. $x_i > 0$ and $y_i = 0$, or
3. $x_i = 0$ and $y_i > 0$.

In any case, $\varphi(q) \in \text{PreTrop}(\{z_i + w_i - 1\}_{i=1}^n)$ for every $q \in Q(\mathcal{T})$.

Recall from Definition 4.4 and Definition 3.21, the augmented gluing equations are of the form

$$ R_k = \prod_{i=1}^{n} (-1)^{c_{k,j}} z_i^{a_{k,j} - c_{k,j}} w_i^{c_{k,j} - b_{k,j}} = 1 $$

and the Q-matching equations are of the form

$$ Q_k = \sum_{j=1}^{n} ((a_{k,j} - c_{k,j})q_{1,j} + (b_{k,j} - a_{k,j})q_{2,j} + (c_{k,j} - b_{k,j})q_{3,j}) = 0 $$

for edges $e_k$ in $\mathcal{T}$ for $1 \leq k \leq n - 1$. The form of $R_k$ in equation 6.1 makes it clear that for a vector $(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n}$ to be in $\text{PreTrop}(R'_k)$ for $1 \leq k \leq n - 1$ it must satisfy:

$$ (x_1, y_1, ..., x_n, y_n) \cdot ((a_{k,1} - c_{k,1}), (c_{k,1} - b_{k,1}), ..., (a_{k,n} - c_{k,n}), (c_{k,n} - b_{k,n})) = 0. $$

(6.2)
When each case is considered in the definition of $\varphi$, we see that if $q \in Q(T)$, then $\varphi(q) \in \text{PreTrop}(\{R'_1, ..., R'_{n-1}\})$.

For example, consider $q = (q_{1,1}, q_{2,1}, q_{3,1}, ..., q_{1,n}, q_{2,n}, q_{3,n}) \in Q(T)$. Let $i \in \{1, ..., n\}$. Assume $q_{2,i} > 0$. Then $(x_i, y_i) = (-q_{2,i}, -q_{2,i})$ in $\varphi(q)$. Then the contribution of $x_i$ and $y_i$ to the inner product of equation 6.2 is

$$-q_{2,i}(a_{k,i} - c_{k,i}) - q_{2,i}(c_{k,i} - b_{k,i}) = (b_{k,i} - a_{k,i})q_{2,i}.$$ 

This is exactly the contribution from $q_{1,i}$, $q_{2,i}$, $q_{3,i}$ in the gluing equation $Q_k = 0$. Therefore, if $q \in Q(T)$, then $\varphi(q) \in \text{PreTrop}(\mathcal{AG})$.

This argument works backwards and so an inverse function can be defined.

The restriction that only one of $q_{i,1}, q_{i,2}, q_{i,3}$ are nonzero for every $q \in Q(T)$ implies that $\varphi$ is linear on each face of $Q(T)$. Therefore $\varphi$ is a piecewise linear homeomorphism from $Q(T)$ to the tropical prevariety $\text{PreTrop}(\mathcal{AG})$. □

We now prove the main result of the thesis. Our proof follows the structure of the proof of Theorem 1.1 in N. Dunfield and S. Garoufalidis’s paper (11).

**Theorem 6.2** Let $(M, T)$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra of $T$. Assume every irreducible component of $\mathcal{D}(T)$ is 1-dimensional. Assume further $q \in \mathbb{Z}^{3n}_{\geq 0}$ is the $Q$-coordinate vector of the surface $S$, and is contained in the relative interior of an isolated ray of $Q(T)$. If the associated properly embedded surface, $\hat{S}$, has nonempty boundary, then $\hat{S}$ is an essential surface in $M$. 
Proof of main theorem: Since \( \varphi : Q(T) \to \text{PreTrop}(\mathcal{A}\mathcal{G}) \) in the proof of Proposition 6.1 is a piecewise linear isomorphism, \( \varphi(q) \) is contained in an isolated ray of the tropical prevariety \( \text{PreTrop}(\mathcal{A}\mathcal{G}) \). Now every irreducible component of \( \mathcal{D}(T) \) is assumed to be 1-dimensional, and so, by Proposition 5.25, \( \varphi(q) \in \text{Trop}(\mathcal{A}\mathcal{G}) \). Then by the Fundamental Theorem of Tropical Geometry (Theorem 5.20), there exist Puiseux series \( p_1, \ldots, p_n \) such that \( \varphi(q) = (v(p_1), \ldots, v(p_n)) \). Notice, since \( q \) is not the origin, \( v(p_i) \neq 0 \) for some \( i \in \{1, \ldots, n\} \).

Therefore, \( S \) is associated to some ideal point \( \xi \) of \( \mathcal{D}(T) \). By assumption, \( \tilde{S} \subseteq M \) has nonempty boundary. Let \( \alpha \) be the boundary slope of \( \tilde{S} \). By Proposition 4.14, if \( \tilde{S} \) is two-sided, then any reduction of \( \tilde{S} \) has nonempty boundary with boundary slope \( \alpha \). Therefore, if \( \tilde{S} \) is two-sided, then \( \tilde{S} \) can be reduced to an essential surface in \( M \) which also has boundary slope \( \alpha \).

Still, we want to show that \( \tilde{S} \) itself is essential. First, we reduce to the case that \( S \) is a vertex surface, and \( \tilde{S} \) is two-sided and connected. If \( S \) is not a vertex surface, then it is some number of copies of the vertex surface for the isolated ray in which it is contained. A spunnormal surface is essential if and only if any number of copies of that spunnormal surface is essential. So we can assume \( S \) is a vertex surface. The definition of an essential non-orientable surface, Definition 1.6, allows us to reduce to the case where \( \tilde{S} \) is two-sided.

As for connectivity, assume \( \tilde{S} \) is not connected, then \( S \) would be the geometric sum of disjoint spunnormal surfaces associated to the components of \( \tilde{S} \). Let \( S_C \) be one of these spunnormal surfaces associated to a component \( C \) of \( \tilde{S} \). Then \( S_C \) is compatible with \( S \). Now \( Q(T) \) is convex when restricted to the \( n \) coordinates representing compatible quad-types of \( S \) and \( S_C \), and \( q \) is
contained in an isolated ray of \( Q(T) \). Therefore, the \( Q \)-coordinate vector \( q_C \in \mathbb{Z}_{\geq 0}^3 \) of \( S_C \) must be in this isolated ray, \( \mathbb{R}_+ \cdot q \). However, \( q_C + w = q \) for some vector \( w \in \mathbb{Z}_{\geq 0}^3 \). This contradicts that \( S \) is the primitive representative of \( \mathbb{R}_+ \cdot q \). So \( \hat{S} \) is connected.

If \( \hat{S} \) is not essential as claimed, there are two possibilities:

1. There is a compressing disk for \( \hat{S} \).
2. \( \hat{S} \) is boundary parallel.

The second case is ruled out since \( \hat{S} \) can be reduced to an essential surface.

So we focus on the first case, and proceed by contradiction.

Before starting, let \( N \) be an end of \( T \) such that \( S \cap (T \setminus N^\bullet) \) maps to \( \hat{S} \) by a homeomorphism \( p_N : (T \setminus N^\bullet) \rightarrow M \) where \( N^\bullet \) is the end \( N \) with the vertex of \( T \). To simplify notation, let \( M' = T \setminus N^\bullet \) and \( \hat{S}' = S \cap M' \). Then \( M' \) is homeomorphic to \( M \) and \( \hat{S}' \) maps to \( \hat{S} \) by this homeomorphism.

Now let \( D \) be a compressing disk for \( \hat{S} \). Then there is a compressing disk \( D' \) for \( \hat{S}' \) which maps to \( D \) by \( p_N \). Compress \( S \) along \( D' \) in \( M' \) to obtain a surface \( S_1 \). Now isotope \( S_1 \) off of \( S \).

Which can be done, because \( S \) was assumed to be two-sided.

Cut \( T \setminus T^{(0)} \) open along \( S \) to obtain a noncompact, orientable 3-manifold with boundary \( P_S \). Here, \( P_S \) is a homeomorphic to \( M' \) cut along \( \hat{S}' \) with annuli corresponding to components of \( \partial M' \setminus \partial \hat{S}' \) removed. Consider the cell decomposition \((P_S, \mathcal{T}_S)\) where \( \mathcal{T}_S \) consists of infinitely many cells formed by cutting the tetrahedra of \( T \) by the normal disks of \( S \). Notice that \( S_1 \) is
disjoint from the boundary of $P_S$ which consists of two copies of $S$. Let $N_S \subseteq P_S$ be $N$ cut along $S$. Then each component of $N_S$ is homeomorphic to $A \times I$ where $A$ is an open annulus, and cells of $T_S$ completely contained in $N_S$ are homeomorphic to $\delta \times I$ where $\delta$ is a triangle. Therefore, $S_1$ intersects each of these cells in normal triangles parallel to the two triangular faces of each cell. See Figure 14.

To simplify notation, let $M_S' = T_S \setminus N_S$. Then $M_S'$ is $M'$ cut along $S$. Let $S_1' = S_1 \cap M_S'$.

Now reduce $S_1'$ inside of $M_S'$ by compressing along compressing disks and removing boundary parallel annuli to obtain an essential surface $S_2'$. Any reduction in $M_S'$ is a reduction in $M'$, and so, by Proposition 4.14, $S_2'$ is a nonempty, essential surface in $M_S'$ with nonempty boundary. If $S_2'$ is not connected, replace it by a connected component with nonempty boundary.

Then we can construct an embedded surface $S_2$ in $P_S$ by attaching the closed boundary curves of half-open annuli to the components of $\partial S_2'$. We require that these attached annuli do not intersect the boundary of $N_S$. Notice that $S_2$ meets each cell of $T_S$ contained in $N_S$ in triangles parallel to the two triangular faces of the cell. Then $S_2$ is disjoint from the boundary $P_S$. See Figure 14.

Now take every cell of $T_S$ which is not completely contained in $N_S$. These form a cell decomposition of a compact, oriented 3-manifold with boundary which is isotopic to $M_S'$. Now by Remark 3.5, we can normalize $S_2$ in this compact cell decomposition. Notice, that since $S_2'$ does not contain any boundary parallel annuli, the normalization process does not push any annuli outside of $M_S'$. The normalization then produces a surface $S_3$ which is normal with
respect to \((P_S, \mathcal{I}_S)\). It could be that \(S_3\) is not isotopic to \(S_2\) if \(M\) is not irreducible. Still, \(S_3\) is topologically the same as \(S_2\), and is normal with respect to \((P_S, \mathcal{I}_S)\). Also, \(S_3\) and the boundary of \(P_S\) are still disjoint since the normalization process does not increase the intersection with edges of \(\mathcal{I}_S\). The possibilities for normal disks of \(S_3\) in \(\mathcal{I}_S\), shown in Figure 14, forces \(S_3\) to be spunnormale with respect to \((M, \mathcal{T})\) and disjoint from \(S\) when the two copies of \(S\) in the boundary of \(P_S\) are glued back together.

Figure 14. The possible cells of \(\mathcal{I}_S\). Dark faces represent normal disks of \(S\). Light normal disks are the possible normal disks for \(S_3\) which cannot intersect dark faces.

Normal disk in cell cut by two triangles

Normal disks in cell cut by quad and two triangles

Normal disks in cell cut by triangles with no quads
(Quad inside can be of any type)

Normal disk in cell cut by two quads
Then $S_3$ is a spunnormal surface with respect to $(M, T)$ and is compatible with $S$. Now $Q(T)$ is convex when restricted to $n$ coordinates representing compatible quad-types of $S$ and $S_3$, and the Q-coordinate vector of $S$ is contained in an isolated ray of $Q(T)$. Therefore, the Q-coordinate vector of $S_3$ must be in the same isolated ray of $Q(T)$. Now $S_3$ was constructed to be two-sided, and therefore must be some number of copies of the two-sided connected surface $S$. However, $S_3$ was also constructed to be connected, and therefore, must be isotopic to $S$. This contradicts that we compressed along $D$ and further reduced $S$ to obtain $S_3$. Therefore, the first case is ruled out and $S$ is essential. $\square$
CHAPTER 7

EXAMPLES

In this chapter, we give sample Python code which defines a function, *the A function*, that finds examples of spunnormal surfaces that satisfy the hypotheses of the main theorem, Theorem 6.2. The function takes in a compact, orientable 3-manifold with torus boundary, $M$, and outputs vertex surfaces which are in an isolated ray of $Q(T)$ where $T$ is the ideal triangulation used by SnapPy for $M$. Theorem 6.2 then shows that these surfaces are essential and their boundary slopes are in $bs(M)$.

The following is the Python code for *the A function* discussed above. It requires the boundary_slopes module from the t3m package, and the cypari module. Everything can be downloaded, or easily found, on the website, http://www.math.uic.edu/t3m/.

```python
from boundary_slopes import *
from cypari.gen import pari

def A(x):
    M=OneCuspedManifold(x)
    M.find_normal_surfaces()
    count=0
    for idx in range(len(M.NormalSurfaces)):
        print 'Spun表面', idx
```

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S = M.NormalSurfaces[idx]

print 'Coefficients:', S.Coefficients

isBad = 0

if 0 in S.Coefficients:
    z = list(S.Coefficients).count(0)
    if z >= 9:
        isBad += 1
        print 'Not Computed'
    else:
        L = list(S.Coefficients)
        p = []
        for i in range(z):
            p += [L.index(0) + i]
            L.remove(0)

        for i in range(len(f(z))):
            for j in range(z):
                S.Quadtypes[p[j]] = f(z)[i][j]
if isGood(S,M)==0:
    isBad=isBad+1
    break
else:
    if isGood(S,M)==0:
        isBad=isBad+1
    if isBad==0:
        print S.info(M)
        count=count+1
        print 'Count: ', count
        print 'Manifold: ', x

The following function is used in the A function to check the dimension of $Q(T)$ for some choice of $n$ coordinates.

def isGood(S, M):
    cols=[3*m+n for m,n in enumerate(S.Quadtypes)]
    T=M.QuadMatrix.rows
    mymatrix=[M.QuadMatrix[i,j] for i in range(T) for j in cols]
    Z=pari.matrix(T,T,mymatrix)
    Z.matsnf()
if list(Z.matsnf()).count(0)==1:
    return 1
else:
    return 0

This final function is used in the A function when generating all of the $3^m$ coordinates that must be checked when $m$ is the number of tetrahedra in which the surface does not have a normal quad.

def f(m):
    if m<=1:
        return [[0],[1],[2]]

    previous=f(m-1)
    result=[ ]
    for vector in previous:
        result = result+[vector + [0], vector + [1],
                         vector +[2]]
    return result

The boundary slopes produced from the A function have been checked for consistency with N. Dunfield’s list of boundary slopes for Montesinos knots up to 10 crossings (8).
Example 7.1 (5₂-knot complement) The following is the output from the A function for the 5₂-knot complement. The quad-types match up with those of Definition 3.2 as follows: \( Q_{13} \) is \( Q_1 \), \( Q_{03} \) is \( Q_2 \), and \( Q_{23} \) is \( Q_3 \).

```python
>>> A('5_2')
slice 0 :  3 positive  3 negative  3 zero
slice 1 :  6 positive  3 negative  0 zero
slice 2 :  0 positive  0 negative  6 zero
DONE. 15 vertices were filtered; 0 were interior.
```

Spun surface 0
Coefficients: [2 2 1]
SpunSurface.
Slope: (-10, -1); Boundary components: 1; Euler characteristic: -2
Incompressible: None

Tet 0: Quad Type Q13, weight 2
Tet 1: Quad Type Q03, weight 2
Tet 2: Quad Type Q23, weight 1
None
Spun surface 1

Coefficients: [1 1 1]

SpunSurface.

Slope: (-4, -1); Boundary components: 1; Euler characteristic: -1
Incompressible: None

Tet 0: Quad Type Q13, weight 1
Tet 1: Quad Type Q13, weight 1
Tet 2: Quad Type Q03, weight 1

None

Spun surface 2

Coefficients: [2 2 1]

SpunSurface.

Slope: (10, 1); Boundary components: 1; Euler characteristic: -2
Incompressible: None

Tet 0: Quad Type Q23, weight 2
Tet 1: Quad Type Q23, weight 2
Tet 2: Quad Type Q23, weight 1

None
Spun surface 3

Coefficients: [1 1 1]

SpunSurface.

Slope: (4, 1); Boundary components: 1; Euler characteristic: -1

Incompressible: None

Tet 0: Quad Type Q03, weight 1
Tet 1: Quad Type Q23, weight 1
Tet 2: Quad Type Q03, weight 1

None

Spun surface 4

Coefficients: [1 2 1]

SpunSurface.

Slope: (0, -1); Boundary components: 1; Euler characteristic: -1

Incompressible: None

Tet 0: Quad Type Q23, weight 1
Tet 1: Quad Type Q13, weight 2
Tet 2: Quad Type Q13, weight 1

None

Spun surface 5
Coefficients:  [2 1 1]

SpunSurface.

Slope: (0, 1); Boundary components: 1; Euler characteristic: -1

Incompressible: None

Tet 0: Quad Type Q03, weight 2
Tet 1: Quad Type Q03, weight 1
Tet 2: Quad Type Q13, weight 1

None

Count = 6

Manifold: 5_2

Notice that the vectors \((1, 1, -1)\) and \((-2, -1, 1)\) found to be in \(\text{Trop}(\overline{\mathcal{R}_1}, \overline{\mathcal{R}_2})\), see Figure 8 and Proposition 5.22, correspond to ‘Spun surface 1’ and ‘Spun surface 5’ respectively from the output of \(A('5_2')\).

The main theorem of this thesis, Theorem 6.2, shows that the six spunnormal surfaces in the output of \(A('5_2')\) are essential, and therefore their boundary slopes are in \(\text{bs}(M)\) for \(M\) the 5_2-knot complement. Then we see from the output that \(\{0, 4, 10\} \subseteq \text{bs}(M)\). From N. Dunfield’s table of boundary slopes for Montesinos knots (8), \(\{0, 4, 10\} = \text{bs}(M)\). In this case the \(A\) function actually found all of \(\text{bs}(M)\) which is rather rare for Montesinos knots up to 10 crossings.
All of these spunnormal surfaces have a normal quad in every tetrahedron, and so these spunnormal surfaces are already known to be essential by N. Dunfield and S. Garoufalidis (11).

Table I gives examples of non-Montesinos knots for which the A function finds boundary slopes of essential spunnormal surfaces with no normal quads in some tetrahedron. As far as the author knows, these boundary slopes are not previously known. These slopes are marked with an asterisks (*) and do not satisfy the criteria of N. Dunfield and S. Garoufalidis (11), but do satisfy the criteria of the main theorem in this thesis, Theorem 6.2. That is, the essential spunnormal surface or spunnormal surfaces with that boundary slope do not have a normal quad in some tetrahedron of the ideal triangulation used by SnapPy.
### TABLE I

A LIST OF BOUNDARY SLOPES OF NON-MONTESINOS KNOTS.

<table>
<thead>
<tr>
<th>Knot</th>
<th>Boundary Slopes</th>
</tr>
</thead>
<tbody>
<tr>
<td>9_{47}</td>
<td>4* 16</td>
</tr>
<tr>
<td>10_{85}</td>
<td>-20 -15 -14 -6* -2</td>
</tr>
<tr>
<td>10_{87}</td>
<td>-14* -10 -9 0 4</td>
</tr>
<tr>
<td>10_{90}</td>
<td>-7 -6* 12 18</td>
</tr>
<tr>
<td>10_{93}</td>
<td>-16* -8* -6* 2</td>
</tr>
<tr>
<td>10_{94}</td>
<td>6 18* 28</td>
</tr>
<tr>
<td>10_{102}</td>
<td>-14* -10* -1 12 18</td>
</tr>
<tr>
<td>10_{103}</td>
<td>-20 -14 -12 -6 8*</td>
</tr>
<tr>
<td>10_{104}</td>
<td>-12* 6 10 14 16</td>
</tr>
<tr>
<td>10_{106}</td>
<td>-14 -8 -6 -4/3 6 14* 18*</td>
</tr>
<tr>
<td>10_{108}</td>
<td>-2 0 11 16*</td>
</tr>
<tr>
<td>10_{110}</td>
<td>-18 -4* 3 14</td>
</tr>
<tr>
<td>10_{112}</td>
<td>-28 -18* -13 6 8 14</td>
</tr>
<tr>
<td>10_{114}</td>
<td>-30 -18 -12 6* 7 14</td>
</tr>
<tr>
<td>10_{116}</td>
<td>-18 -14* -13 -12 -9 8 14</td>
</tr>
<tr>
<td>10_{117}</td>
<td>-14 7 8* 12 18</td>
</tr>
<tr>
<td>10_{118}</td>
<td>-26 -16 -12* 0 12* 16 26</td>
</tr>
<tr>
<td>10_{119}</td>
<td>-30 -7 -6* -4 12 18 24</td>
</tr>
<tr>
<td>10_{123}</td>
<td>-12* 12*</td>
</tr>
</tbody>
</table>

* boundary slopes from essential spunnormal surfaces with no normal quads in a tetrahedron
CITED LITERATURE


Andrew Brasile

Curriculum Vitae

Education

2008–2013  **PhD in Pure Mathematics**, University of Illinois at Chicago, Chicago, IL.

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**Presentations**

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March 23, 2012  **Cooperative learning workshops: Discussion on organization and methods**, *2012 Chicago Symposium Series*, Northwestern University, Chicago, IL.

September 28, 2011  **Paradoxical decompositions and hyperbolic 3-manifolds**, *Graduate Geometry, Topology and Dynamics Seminar*, University of Illinois at Chicago, Chicago, IL.

April 20, 2011  **The Thurston compactification**, *Graduate Geometry, Topology and Dynamics Seminar*, University of Illinois at Chicago, Chicago, IL.

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